Take it to the limit, one more time

Leo Livshits

In the previous handout “Rigorous definition of Continuity and Limits of Functions” we have introduced a concept of the limit of a function on the real line in such a way that it is immediately obvious what the definition shall be for functions \( f : D \rightarrow \mathbb{R}^m \) with \( D \subset \mathbb{R}^n \). See for yourself.

1 Problem. Suppose \([W_n] \) is obtained from sequence \([V_m] \) in \( \mathbb{R}^k \) by interspersing a (possibly infinite) number of \( M \)'s \( (M \in \mathbb{R}^k) \) among the terms of \([V_m] \). Prove:
\[
[W_n] \rightarrow M \iff [V_n] \rightarrow M
\]

Hint: Prove this in the case \( k = 1 \) and then make judicious use of one of the equivalent definitions of sequential limits in \( \mathbb{R}^k \).

2 Problem. Given \( A \in D \subset \mathbb{R}^n \), and \( f : D \rightarrow \mathbb{R}^m \), prove that the following are equivalent:

1. \( f \) is continuous at \( A \)
2. \( \lim_{W \rightarrow A} f(W) = f(A) \)

Caution: Part 2 only states something about sequences none of whose terms are \( A \), while Part 1 involves no such restriction. You will want to use Problem 1.

3 Problem. Given \( A \in D \subset \mathbb{R}^n \), and \( f : D \rightarrow \mathbb{R}^m \), prove that the following are equivalent:

1. \( \lim_{W \rightarrow A} f(W) = L \)
2. Function \( \hat{f}(U) \) defined by
\[
\hat{f}(U) = \begin{cases} 
  f(U), & \text{if } U \neq A \\
  L, & \text{if } U = A 
\end{cases} 
\]
\((U \in D)\), is continuous at \( A \).

Caution: Part 1 only states something about sequences none of whose terms are \( A \), while Part 2 involves no such restriction. You will want to use Problem 1.

Here is a theorem that summarize the “limit rules”. You can take it for granted, since the proofs easily reduce to the one-dimensional case, via the fact that sequential convergence in \( \mathbb{R}^k \) is equivalent to “entry-wise” convergence.
1 Theorem. Here $A \in D \subset \mathbb{R}^n$, and $f, g : D \rightarrow \mathbb{R}^m$.

1. $\lim_{W \rightarrow A} f(W) = L \iff \lim_{W \rightarrow A} \|f(W) - L\| = 0$.

2. If $f$ is constantly $C$, then $\lim_{W \rightarrow A} f(W) = C$.

3. Suppose $\lim_{W \rightarrow A} f(W) = L$ and $\lim_{W \rightarrow A} g(W) = M$. Then
   
   (a) $\lim_{W \rightarrow A} (\alpha f + \beta g)(W) = \alpha L + \beta M$
   
   (b) $\lim_{W \rightarrow A} (f \cdot g)(W) = L \cdot M$
   
   (c) $\lim_{W \rightarrow A} ||f|| (W) = ||L||$
   
   (d) If $m = 3$ then $\lim_{W \rightarrow A} (f \times g)(W) = L \times M$
   
   (e) If $m = 2$ or $4$ then $\lim_{W \rightarrow A} (f g)(W) = LM$ (with multiplication being complex/quaternion multiplication respectively)
   
   (f) If $m = 2$ or $4$ then $\lim_{W \rightarrow A} (f^*)(W) = L^*$ (with conjugation being complex/quaternion conjugation respectively)
   
   (g) If $m = 2$ or $4$ and $M \neq O_m$ then $\lim_{W \rightarrow A} (f / g)(W) = L / M$ (with reciprocation being complex/quaternion reciprocation respectively)
   
   (h) If $h$ is continuous at $L$ then $\lim_{W \rightarrow A} (h \circ f)(W) = h(L)$. 