Bolzano-Weierstrass for Sequences in Higher Dimensions

Leo Livshits
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1 Definition. Given a subset $\Omega$ of a metric space $(X, \rho)$, we say that $\Omega$ is $(\rho)$-bounded if there exists some positive real number $M$ such that
\[ \forall a, b \in \Omega : \rho(a, b) \leq M. \]
In other words: if the set $\{ \rho(a, b) \mid a, b \in \Omega \}$ is a bounded subset of $\mathbb{R}$. In that case the least upper bound of this set is said to be the diameter of $\Omega$.

1 Problem.

1. Suppose $d$ is a norm metric on $\mathbb{R}^n$ generated by norm $\| \|$ . Show that the following are equivalent for a subset $\Omega$ of $\mathbb{R}^n$:
   
   (a) $\Omega$ is $d$-bounded;
   
   (b) $\Omega$ is $\| \|$-bounded, in the sense that there is some number $M$ such that
   \[ \|u\| < M \text{ for all } u \in \Omega; \]
   which is the same as saying that the set $\{ \|u\| \mid u \in S \}$ is bounded (in $\mathbb{R}$), or equivalently that
   \[ \Omega \subset B^\| \|_M (0). \]
   
   (A minute’s thought will convince you that when $n = 1$ this reflects your previous vision of “bounded subsets of $\mathbb{R}$”.)

2. Show that concept of a bounded set in $\mathbb{R}^n$ is norm-independent; in other words, given any norms $\phi$ and $\psi$ on $\mathbb{R}^n$, a set $S$ is $\phi$-bounded if and only if it is $\psi$-bounded. This proof should be quite short, once you find the right result to quote from your notes.

Now we will just say that a set is (norm-)bounded in $\mathbb{R}^n$ if it is bounded with respect to some norm, it does not matter which.
2 Definition. Let us define coordinate projections $\Pi_i : \mathbb{R}^n \rightarrow \mathbb{R}$ by the formula

$$
\Pi_i \left( \begin{array}{c}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_n \\
\end{array} \right) = x_i
$$

3 Notation. Given $f : X \rightarrow Y$ and a subset $Z$ of $X$, we write $f^{-1}(Z)$ for the set

$$
\{ f(x) \mid x \in Z \}
$$

2 Problem.

1. Prove that if $d$ is a norm metric on $\mathbb{R}^n$ then each $\Pi_i : (\mathbb{R}^n, d) \rightarrow \mathbb{R}$ is a continuous function. This proof should be very short, once you choose a convenient definition of continuity and quote a relevant result.

2. Prove that the following are equivalent for a subset $\Omega$ of $\mathbb{R}^n$:

   (a) $\Omega$ is bounded
   (b) $\Pi_i^{-1}(\Omega)$ is bounded for every $i$.

3. Use part 1 to prove that

$$
\{ (x, y, z, w) \mid x^4 - y^6 + e^{|z|} + |w| = 11 \}
$$

is not bounded in $\mathbb{R}^4$, but

$$
\{ (x, y, z, w) \mid x^4 + y^6 + |z| + e^{|w|} = 11 \}
$$

is.

4 Definition. A sequence in a metric space $(X, \mu)$ is said to be bounded if its range is a bounded set.

3 Problem. B-W for sequences in $(\mathbb{R}^n, \| \|)$.

1. B-W for sequences in $(\mathbb{R}^2, \| \|)$ states:

   Every (norm-)bounded sequence in $\mathbb{R}^2$ has a convergent subsequence.

   Your task is to fill in the missing details in the following proof outline.
Proof. Let us write “DPFJ” for “Deduce, providing full justification.”
Suppose \([a_n, b_n]\) is a bounded sequence.
DPFJ that sequence \([a_n]\) is bounded in \(\mathbb{R}\).
DPFJ that \([a_n]\) has a convergent subsequence \([a_{n_k}]\); say \([a_{n_k}] \longrightarrow \alpha\).
Then \([(a_{n_k}, b_{n_k})]\) is a subsequence of \([(a_n, b_n)]\).
DPFJ that \([b_{n_k}]\) is a bounded sequence.
DPFJ that it has a convergent subsequence \([b_{n_{km}}]\); say \([b_{n_{km}}] \longrightarrow \beta\).
Then \([(a_{n_{km}}, b_{n_{km}})]\) is a subsequence of \([(a_n, b_n)]\).
DPFJ that \([a_{n_{km}}]\) converges to \(\alpha\). As we know, \([b_{n_{km}}] \longrightarrow \beta\).
DPFJ that \([(a_{n_{km}}, b_{n_{km}})] \longrightarrow (\alpha, \beta)\), and you are done. \( \square \)

2. Use induction to prove B-W theorem for sequences in \(\mathbb{R}^n\).

1 Extra Credit Problem. This problem has the same weight as the other problems.

1. Suppose that \(d\) is a metric on \(X\), and \(\varphi : X \longrightarrow X\) in an injection. Show that

\[ \rho(a, b) \overset{\text{def}}{=} d(\varphi(a), \varphi(b)) \]

defines a metric \(\rho\) on \(X\). Why is the assumption of injectivity essential?

2. Find a metric \(\mu\) on \(\mathbb{R}\) such that the interval \([0, 1]\) is NOT \(\mu\)-bounded.

3. Find a metric \(\gamma\) on \(\mathbb{R}^2\) such that the set \(\{ (x, y) \mid x^2 + y^2 = 1 \}\) is NOT \(\gamma\)-bounded.

4. Find a metric \(\beta\) on \(\mathbb{R}\) such that B-W theorem fails for \((\mathbb{R}, \beta)\).

5. Find a metric \(\delta\) on \(\mathbb{R}^2\) such that B-W theorem fails for \((\mathbb{R}^2, \delta)\).