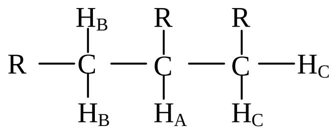
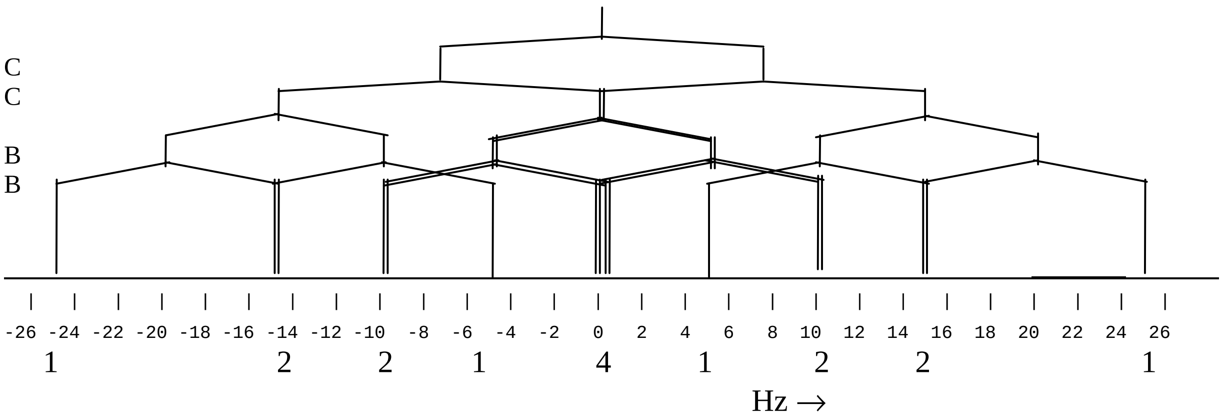
CH342 Handin Homework 3

1. Show the spin-spin splitting pattern for nucleus A in the following molecular fragment. J_{AB} = 10 Hz and J_{AC} = 15 Hz. Indicate the relative intensities.



partner:



2. Normalize the wave function for the ground state of the harmonic oscillator: $\Psi(x) = A\, e^{-\frac{1}{2}\alpha^2 x^2}$.(Don't use Eq. 24.4.44)

Answer: The plan is to use the normalization integral, $\int_{-\infty}^{\infty} \Psi^* \Psi \, dx = 1$, to find the normalization constant N.

Remember that $(e^x)^2 = e^{2x}$. Substitution of the wave function into the normalization integral gives:

$$N^2 \int_{-\infty}^{\infty} e^{-\alpha^2 x^2} dx = 1 \qquad \text{with} \quad \alpha^2 = m\omega_o/\hbar$$

This integrand is even about x = 0. Using the table in the Appendix: $\int_0^{\infty} e^{-ax^2} dx = \frac{1}{2} (\pi/a)^{\frac{1}{2}}$:

$$N^2\, 2 \int_0^{\infty} e^{-\alpha^2 x^2} dx = N^2 (\pi/\alpha^2)^{\frac{1}{2}} = 1$$

Solving for the normalization constant gives: $N = \left(\frac{\alpha^2}{\pi}\right)^{\frac{1}{4}} = \left(\frac{m\omega_o}{\pi\hbar}\right)^{\frac{1}{4}}$

The complete wave function is then: $\Psi_o = \left(\frac{\alpha^2}{\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\alpha^2 x^2} = \left(\frac{m\omega_o}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega_o}{2\hbar} x^2}$

3. Find the average position of the harmonic oscillator in its ground state. Hint: find the expectation value of x for the wave function given in the last problem.

Answer: The given Gaussian wave function is normalized, $\int_{-\infty}^{\infty} \Psi^2 \, dx = 1$, and real, $\Psi^*\Psi = \Psi^2$. The average position is zero for a symmetrical distribution because x is odd and Ψ^2 is even and the integral is taken over all space:

$$\langle x\rangle = \underbrace{\int_{-\infty}^{\infty} \Psi^* \, x \, \Psi \, dx}_{normalized} = \underbrace{\int_{-\infty}^{\infty} x \, \Psi^2 \, dx}_{real} = N^2 \int_{-\infty}^{\infty} \underbrace{x}_{odd} \, \underbrace{e^{-\alpha^2 x^2}}_{even} \, dx = 0$$

4. Show that $e^{-i\alpha x}$ is an eigenfunction of both the kinetic energy and momentum operators.

Answer: The plan is to note that the operator for momentum is $\hat{p}_x = (\hbar/i)(d/dx)$. The operator for kinetic energy in the x-direction is $\hat{E}_k = -(\hbar^2/2m)(d^2/dx^2)$.

If $\hat{o}\Psi = o\Psi$, with o a constant, then Ψ is an eigenfunction. For the momentum:

$$\hat{p}_x \Psi = \frac{\hbar}{i}\frac{d}{dx} e^{-i\alpha x} = -\hbar \alpha \, e^{-i\alpha x} = -\hbar\alpha \Psi$$

The wave function is an eigenfunction of the momentum operator, so every observation of the momentum will give the same result, $-\hbar\alpha$. By Postulates III and IV, if the wave function is an eigenfunction of the operator representing the given observable, then repeated measurements will give identical results.

For the kinetic energy:

$$\hat{E}_k\Psi = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\Psi = \frac{\hbar^2\alpha^2}{2m} e^{-i\alpha x} = \frac{\hbar^2\alpha^2}{2m}\Psi$$

The wave function is an eigenfunction of the kinetic operator, so every observation of the kinetic energy will give the same result, $\hbar^2\alpha^2/2m$.