Particle in a Box

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$$\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\Psi + V(x)\Psi = E\,\Psi$$

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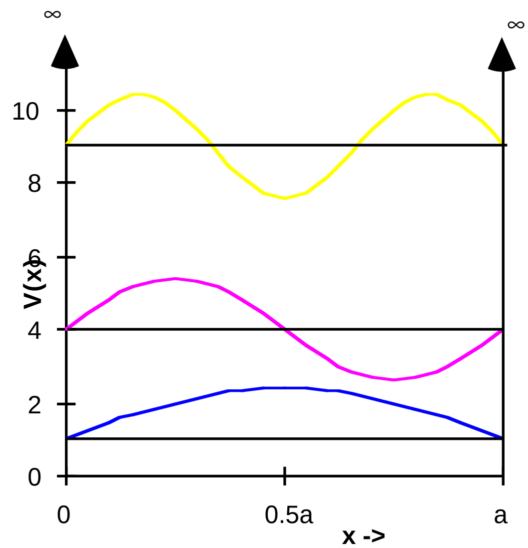
for $0 \leq x \leq a$



A sin kx : $\frac{d\Psi}{dx} = A\,k\cos kx$ $\quad \frac{d^2\Psi}{dx^2} = -A\,k^2 \sin kx$

B cos kx : $\frac{d\Psi}{dx} = -B\,k\sin kx$ $\quad \frac{d^2\Psi}{dx^2} = -B\,k^2 \cos kx$

<u>LHS</u>: <u>RHS</u>:

$$\frac{-\hbar^2}{2m}(-A\,k^2 \sin kx) = E\ A \sin kx$$

$E = \frac{\hbar^2 k^2}{2m}$ $\qquad k = \frac{(2mE)^{1/2}}{\hbar}$

<u>General Solution:</u> $\Psi(x) = A \sin kx + B \cos kx$

<u>Boundry Conditions:</u> at x=0 $\Psi(x) = 0$ $\quad \cos(0) = 1$ so B = 0

at x=a $\Psi(x) = 0$

$\Psi(a) = A \sin ka$ so $k = n\pi/a$

n = 1 $\quad kx = \frac{n\pi x}{a}$ $\quad$ at a: $ka = \pi$

n = 2 $\quad ka = 2\pi$

$$E = \frac{n^2\hbar^2\pi^2}{2ma^2} = \frac{n^2h^2}{8ma^2}$$

Normalization $\int_{-\infty}^{\infty} \Psi^2 \, dx = 1$

$$= A^2 \int_0^a \sin^2\left(\frac{n\pi x}{a}\right) dx$$ for particle in the box.

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let $y = \frac{n\pi x}{a}$ then $\frac{dy}{dx} = \frac{n\pi}{a}$ and rearranging gives $x = \frac{a}{n\pi} y$ and $dx = \frac{a}{n\pi} dy$
and when x = 0 to a, then y = 0 to nπ.

Substituting to change variables gives:

$$A^2 \left(\frac{a}{n\pi}\right) \int_0^{n\pi} \sin^2(y) \, dy$$

looking up the integral in the CRC:

$$\int \sin^2(x) \, dx = \frac{-1}{2} \sin x \cos x + \frac{1}{2} x$$

substituting gives:

$$A^2 \left(\frac{a}{n\pi}\right) \left(\frac{-1}{2} \sin y \cos y + \frac{1}{2} y \right)\Bigg|_0^{n\pi}$$

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note that sin(0) = 0 and sin(nπ)=0.
Evaluating the result at the endpoints gives:

$$A^2 \frac{a}{2} = 1$$

$$A = \left(\frac{2}{a}\right)^{1/2}$$
