Problem Set 2
Solutions

Chapter 6

6.8 Firm 1's profit maximization problem is:

$$\max_{q_1} (a - bq_1 - d\overline{q}_2)q_1 - cq_1$$

$$FOC: a - 2bq_1 - d\overline{q}_2 = c \Rightarrow q_1 = \frac{a - c - d\overline{q}_2}{2b}$$

The numerator in $\frac{a-c-d\overline{q}_2}{2b}$ is nonnegative so long as $\overline{q}_2 \le \frac{a-c}{d}$. Thus firm 1's best response function is given by

$$R_1(Q_2) = \begin{cases} \frac{a-c-d\overline{q}_2}{2b} & \text{if } q_2 \le \frac{a-c}{d} \\ 0 & \text{if } q_2 > \frac{a-c}{d} \end{cases}$$

6.9 Firm 2's best response function is given by

$$R_2(Q_1) = \begin{cases} \frac{a-c-dq_1}{2b} & \text{if } q_1 \le \frac{a-c}{d} \\ 0 & \text{if } q_1 > \frac{a-c}{d} \end{cases}$$

Hence in equilibrium,

$$q_1 = \frac{a - c - d\frac{a-c-dq_1}{2b}}{2b}$$

$$\Rightarrow 2bq_1 = a - c - d\frac{a-c-dq_1}{2b} \Rightarrow (2bq_1 - a + c)2b = -ad + cd + d^2 q_1$$

$$\Rightarrow 4b^2 q_1 - d^2 q_1 = 2ab - 2bc - ad + cd \Rightarrow q_1 = \frac{(a-c)(2b-d)}{4b^2 - d^2}$$

$$\Rightarrow q_1 = \frac{a-c}{2b+d} = q_2$$

6.10 The cartel's maximization problem is

$$\max_{q_1, q_2} (a - bq_1 - dq_2)q_1 - cq_1 + (a - bq_2 - dq_1)q_2 - cq_2$$

FOC_1 (derivative with respect to q_1) : $a - 2bq_1 - 2dq_2 - c = 0$
FOC_2 (derivative with respect to q_2) : $a - 2bq_2 - 2dq_1 - c = 0$

$$\Rightarrow q_1 = \frac{a - c - 2dq_2}{2b} \text{ and } q_2 = \frac{a - c - 2dq_1}{2b}$$

By substituting, we get

$$q_1 = \frac{a - c - 2d\,\frac{a - c - 2dq_1}{2b}}{2b} \Rightarrow 2bq_1 = a - c - d\,\frac{a - c - 2dq_1}{b}$$

$$2b^2 q_1 - 2d^2 q_1 = ab - bc - ad + cd \Rightarrow q_1 = \frac{(a - c)}{2(b + d)} = q_2$$

6.11 The cartel produces less than total production under the Cournot Nash equilibrium, as long as $\frac{(a-c)}{(b+d)} < \frac{2(a-c)}{2b+d}$. Since $a > c$, this is equivalent to $\frac{1}{b+d} < \frac{2}{2b+d}$, which is equivalent to $2b + d < 2b + 2d$, which is clearly true. The two firm's products are coarse substitutes. Hence, increased production by one firm reduces the price of the other firm's product. A cartel would curb production relative to the competitive case so as to support higher prices for both products.

6.12 The ratio of $\frac{2b+d}{2(b+d)} > \frac{3}{4}$ if $6(b + d) < 4(2b + d)$, which is equivalent to $6b + 6d < 8b + 4d$, which is equivalent to $2d < 2b$, which is the case if $d < b$. If $d < b$, then the price faced by each firm is more affected by its own production than by the production of the rival. When this is the case, the optimal cartel quantity is higher than in the homogeneous good case, since the cost of overproduction is mitigated. Thus, the ratio between cartel output to Nash equilibrium output is higher.

Chapter 7

7.9

Strategy for firm 1: extraction in period 1 = c, extraction in period 2 = $25 - c$

Strategy for firm 2: extraction in period 1 = d, extraction in period 2 = $25 - d$

Total profits of firm 1 = $(100 - c - d)c + (50 + c + d)(25 - c)$

Total profits of firm 2 = $(100 - c - d)d + (50 + c + d)(25 - d)$

7.10

The profit maximization problem of firm 1 is:

$$\max_{c}\ (100 - c - d)c + (50 + c + d)(25 - c)$$

$$FOC:\ 100 - 2c - d - 50 + 25 - 2c - d = 0$$

$$\Rightarrow 75 - 4c - 2d = 0 \Rightarrow c = \tfrac{75-2d}{4}$$

If firm 2 produces 7.5 units of output in the first period, then firm 1 should produce $\frac{75-15}{4} = 15$.

7.11

Firm 2's best response function, by the same logic, is $d = \frac{75-2c}{4}$. Hence, in Nash equilibrium,

$$c = \frac{75 - 2\frac{75-2c}{4}}{4} \Rightarrow 6c = 75 \Rightarrow c = 12.5 = d$$

Hence, the equilibrium extraction levels are 12.5 and 12.5 in periods 1 and 2 for both firms. In both periods, the market price is $100 - (12.5 + 12.5) = 75$.

7.14

The cartel problem, assuming that both firms produce the same quantities c in the first period, is:

$$\max_{c}\ 2500 + 200c - 8c^2$$

$$FOC:\ 200 = 16c \Rightarrow c = 12.5$$

Hence, the cartel solution leads to the same amount of extraction as the competitive case. In the Cournot model of Chapter 6, the cartel solution involved lower production than the competitive case, because in the latter case, each firm ignored the cost of higher production on the rival (in terms of lower prices). In the resource extraction problem, the competitive firm cannot ignore the cost of higher production since higher production in the first period leads to lower profits in the second period. In this problem, the two firms impose externalities on each other, but the externalities cancel out across the two periods.

Additional Exercises

1. There are an infinite number of Nash equilibria of this game: all combinations of s_1 and s_2 such that $s_1 + s_2 = \$1$ are Nash equilibria. Written formally, NE = $\{(s_1^*, s_2^*): s_1^* + s_2^* = 1\}$. If the shares add up to \$1, whether they are equal shares or not, neither player has an incentive either to raise or to lower his share.

2. In equilibrium, 6 firms locate downtown and 4 locate in the suburbs. Each firm earns a profit of 44.

3. (a) Players: N farmers

Strategies: Player i chooses number of goats g_i s.t. $g_i \geq 0$

Payoffs: Player i's payoff is $\pi_i = g_i \text{v}(g_i + G_{-i}) - cg_i$

(b) In a Nash equilibrium, farmer i solves

$$\max_{g_i} \ \pi_i(g_i, G_{-i}) = g_i \text{ v}(g_i, G_{-i}) - cg_i \tag{1}$$

The first-order condition in a Nash equilibrium is

$$\text{v}(g_i^* + G_{-i}^*) + g_i^* \text{ v}'(g_i^* + G_{-i}^*) - c = 0 \tag{2}$$

(c) The social optimum is characterized by

$$\max_{G} \ G \text{ v}(G) - cG \tag{3}$$

The first-order condition in the social optimum is

$$\text{v}(G^{**}) + G^{**} \text{ v}'(G^{**}) - c = 0 \tag{4}$$

(d) Show that $G^* > G^{**}$ by contradiction. In other words, show that if $G^* \leq G^{**}$ then a contradiction arises that cannot be true.

First, sum equation (2) over all N farmers and divide by N. This yields

$$\text{v}(G^*) + \frac{1}{N} G^* \text{ v}'(G^*) - c = 0 \tag{5}$$

Compare equation (5) and equation (4). Note that if $G^* \leq G^{**}$ then the following three conditions hold:

(i) $\text{v}(G^*) \geq \text{v}(G^{**})$ because $\text{v}' < 0$

(ii) $0 \geq \text{v}'(G^*) \geq \text{v}'(G^{**})$, where the first inequality holds because $\text{v}' < 0$ and the second inequality holds because $\text{v}'' < 0$

(iii) $\frac{1}{N} G^* < G^{**}$ [and notice that these terms are mulitplying a negative quantity in both equation (5) and equation (4)]

Taken together, (i), (ii), and (iii) imply that the left hand side of equation (5) is strictly greater than the left hand side of equation (4). But this cannot be true if both are equal to zero. Hence, a contradiction arises, and it must be the case that $G^* > G^{**}$.