Pictures of Ultrametric Spaces, the p-adic Numbers, and Valued Fields

Jan E. Holly

1. INTRODUCTION. When studying a metric space, it is valuable to have a mental picture that displays distance accurately. When the space is $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$, we usually form such a picture by imagining points on the "number line". When the space is $X = \mathbb{Z}^2$, $\mathbb{Q}^2$, $\mathbb{R}^2$, or $\mathbb{C}$ we use a planar picture in which nonempty discs (sets of the form $\{x \in X : d(x, b) \le \gamma\}$ or $\{x \in X : d(x, b) < \beta\}$, with metric d, point $b \in X$, γ nonnegative, and β positive) can be drawn on paper with a circular shape, and the triangle inequality is demonstrated by drawings of triangles. This assumes that the standard metric is in use, but even with a slightly different metric, the planar picture might still be useful; discs might be diamond-shaped instead of round, for example.

However, we find that if the space is non-Archimedean (i.e., if it is an ultrametric space), then the usual pictures lose their utility. An *ultrametric space* X is a metric space in which the metric satisfies the *strong triangle inequality*

$$d(x, z) \le \max\{d(x, y), d(y, z)\}$$

for all $x, y, z \in X$. In such a space, various interesting things happen: Triangles are always isoceles with the unequal side (if any) being shortest; every point in a given disc is a center of that disc; two discs can intersect only by having one completely contained in the other. If we imagine an ultrametric space as having its points on a line or in a plane, we cannot appeal to our usual intuition for distance. Instead, it is useful to have a new framework for visualizing the ultrametric space, and we propose using a different picture—that of a *tree*.

One well-known ultrametric space is $\mathbb{Q}_p$, the field of p-adic numbers, and it is our main example in Section 2. The tree picture serves for other valued fields as well, and we discuss this in Section 4, after looking more closely at triangles and discs in Section 3. Before embarking on a study of these particular fields, however, it is instructive to discuss another example.

Consider the set of all species in the animal kingdom, and look at the tree of classification for these species in Figure 1.

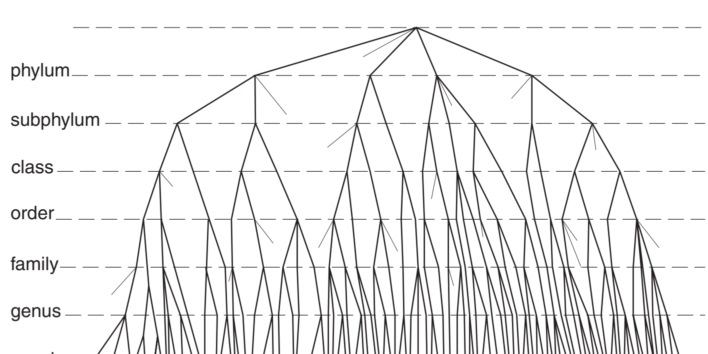

Figure 1.

Intuitively, we might think of the "distance" between two species b and c as being a measure of the differences between their traits, the distance between b and c being small if they belong to the same genus, but large if they share only the same phylum. So a good measure of distance between b and c is the height in the (inverted) tree to which one must climb in traversing a path from b to c. By assigning numerical values to the different levels (0 to "species", 1 to "genus", 2 to "family", etc.), the distance function (call it d) described in this way becomes an ultrametric: $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ for any three species x, y, and z.

Thus, the picture in Figure 1 gives an easy way to see distance between two species, and it turns out that for *any* ultrametric space, such a tree can be drawn. For further interesting examples of ultrametric spaces in the context of trees, see [**5**].

2. THE *p*-ADIC METRIC. The field $\mathbb{Q}_p$ of p-adic numbers presents a particular challenge to the intuition. It is an extension of $\mathbb{Q}$, so the temptation is to imagine $\mathbb{Q}$, first, as a subset of the real line. However, this makes life difficult when trying to place the additional points of $\mathbb{Q}_p$, because p-adic distances turn out to have no relation to distances on the line. Thus, it is necessary to start from scratch in drawing a picture of $\mathbb{Q}_p$. One approach is given in [**1**], and we present another. Our approach is general enough to apply to other ultrametric spaces as well as to valued fields.

We do this by first considering $\mathbb{Z}$ and $\mathbb{Q}$, developing the appropriate picture, and then extending it for $\mathbb{Q}_p$, introducing and formally defining $\mathbb{Q}_p$ in the process. For now, it suffices to know that $\mathbb{Q}_p$ contains all rational numbers as well as some new elements that we introduce later in the section.

What do $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{Q}_p$ look like as metric spaces when using the p-adic metric? Roughly speaking, the p-adic metric, for a fixed prime p, says that two points are "close" if their difference is divisible by a large positive power of p. When using the 7-adic metric, for example, 2 and 51 are closer together than are 2 and 3, because $51 - 2 = 49 = 7^2$ while $3 - 2 = 1 = 7^0$. A precise definition follows.

Throughout, p is a fixed prime.

Definitions. Given a number $b \in \mathbb{Q}$, write $b = rp^n/s$ for $r, s, n \in \mathbb{Z}$, where p divides neither r nor s. Then $\mathrm{ord}_p(b) := n$. The *p-adic norm* on $\mathbb{Q}$ is defined by

$$|b|_p := \frac{1}{p^{\mathrm{ord}_p b}},$$

and the *p-adic metric* on $\mathbb{Q}$ is the metric induced by the p-adic norm, so $|x - y|_p$ is the p-adic distance between x and y.

One checks that $|\ |_p$ is indeed a norm, and that the p-adic metric is an ultrametric.

To start a picture, we focus on some nice members of $\mathbb{Q}$, the integers. If x is an integer, notice that $\mathrm{ord}_p(x)$ is non-negative and $|x|_p \leq 1$. Large positive powers of p are p-adically small: $|p^n|_p = 1/p^n$.

Now suppose $p = 3$, and consider $\mathbb{Z}$ as an ultrametric space with the 3-adic metric. The distance between two elements x and y can be at most 1, corresponding to $\mathrm{ord}_3(x - y) = 0$; the next distance possible is $1/3$, corresponding to $\mathrm{ord}_3(x - y) = 1$; all possible distances are 1, $1/3$, $1/3^2$, $1/3^3$, etc. With this in mind, it is possible to start drawing the tree for $\mathbb{Z}$, as in Figure 2. In this figure, the levels for distance are labeled in the same way as discussed for the tree of classification of species, and corresponding "ord_3" labels are also given (to be used in Section 4). The 3-adic distance between any two integers b and c is the height, as labeled in the distance column, to which one must climb in traversing a path from b's leaf to c's leaf.

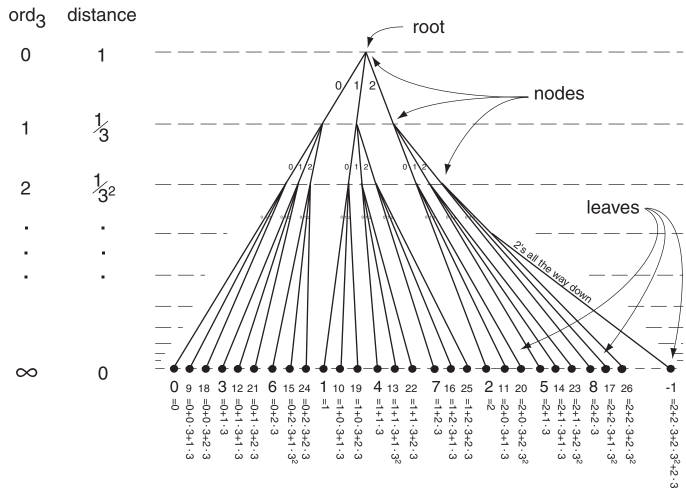

Figure 2.

A few observations can be made about the structure of the tree in Figure 2. The first is that there are infinitely many "distance" levels in the tree. The second is that, as for any metric space, it is physically possible to highlight only finitely many elements; in Figure 2, the only negative integer shown is -1, even though all of the negative integers are really in the tree (and the reader may wish to try adding some of them to the drawing, based upon their 3-adic distances from the other integers). Practically speaking, it does not matter what angles the branches make with one another, or even whether a given integer appears to the left or right of another given integer, as long as all of the heights in climbing between integers are correct. The third observation is that, with infinitely many integers in the entire tree, any given integer has distance 1 from infinitely many other integers. The same holds for each of the distances $1/3$, $1/3^2$, etc.

It turns out that the tree we have drawn is intimately related to p-adic expansions of numbers, so it is important to know ([**2**, p. 66], [**4**, p. 13]) that given a prime p and the p-adic metric, every rational number can be written as a series of the form

$$\sum_{k=n}^{\infty} b_k p^k \quad \text{for some } n \in \mathbb{Z}, \text{ and } b_k \in \{0, 1, \ldots, p-1\} \text{ for each } k \geq n.$$

This series is called the *p-adic expansion* of the number. Positive integers have finite expansions, while negative integers and many non-integers are represented by infinite series.

Although these infinite series may seem strange at first, a fact such as

$$-1 = 2 + 2 \cdot 3 + 2 \cdot 3^2 + 2 \cdot 3^3 + \cdots \qquad \text{under the } p\text{-adic metric}$$

is no more strange than

$$1/3 = 0.333\ldots = 3(0.1) + 3(0.1)^2 + 3(0.1)^3 + \cdots \qquad \text{under the standard metric.}$$

Each of these series converges by the usual definition of convergence, using the stated metric. For example,

$$\begin{aligned}\lim_{n\to\infty} &|(2+2\cdot 3+2\cdot 3^2+\cdots+2\cdot 3^n)-(-1)|_3\\ &=\lim_{n\to\infty}|3+2\cdot 3+2\cdot 3^2+\cdots+2\cdot 3^n|_3\\ &=\lim_{n\to\infty}|3^{n+1}|_3\\ &=0\end{aligned}$$

tells us that $-1 = 2 + 2\cdot 3 + 2\cdot 3^2 + \cdots$. Similarly, $-2 = 1 + 2\cdot 3 + 2\cdot 3^2 + \cdots$, etc.

These p-adic expansions can help us understand the tree for $\mathbb{Z}$. Labeling the branches underneath each node with "0", "1", and "2" as in Figure 2, one sees that a trip from the root down to an integer is really a sequence of choices of the coefficients in the 3-adic expansion of the integer. In addition, the p-adic norm of an integer can be seen: It is the distance label at the level at which the path from the root down to that integer branches away from the path down to 0. Thus, an integer is divisible by a large power of 3 if the path from the root down to that integer starts by following much of the path down to 0.

It doesn't take long to see self-similarity appearing in the tree. From each node, exactly three branches descend, and each branch represents a choice of coefficient for 3-adic expansions of integers.

New integers can be added to the drawing by considering their 3-adic expansions and beginning the corresponding trip down from the root, drawing a new branch at the level where the new integer's 3-adic expansion differs from those of all the integers already shown. One can think of a tree for $\mathbb{Z}$ as being the result of an infinite process of adding integers in this way. In fact, it can be proved [**3**] that for any given prime p, there is such a tree accurately portraying $\mathbb{Z}$ as a p-adic metric space, and this tree is unique up to ordering of the branches descending out of each node.

What about the leaves at the bottom of the tree? We must be careful because the "final" tree is the limit of an infinite process, and a complete path from a root to a leaf descends through infinitely many levels. At the end of each descending path that determines the p-adic expansion of an integer, there is a leaf. However, not every infinite p-adic expansion $b_0 + b_1 p + b_2 p^2 + \cdots$ describes an integer! It would be interesting to allow *all* infinite branches to have leaves, and this is indeed what we do to form $\mathbb{Q}_p$, after first expanding $\mathbb{Z}$'s tree to a tree for $\mathbb{Q}$.

To expand the tree for $\mathbb{Z}$ to include all rational numbers, look at p-adic expansions again. As before, consider $p = 3$. Suppose we want to add the number $1/3$ to the tree for $\mathbb{Z}$. Since $1/3$ is at distance 3 from each element of $\mathbb{Z}$, the tree must be expanded to include a new, higher level. At that level, a node is placed, from which descend three branches: one to the root of $\mathbb{Z}$'s tree, one toward a leaf for $1/3$ (and ultimately serving the subtree for all rational numbers whose 3-adic expansions begin "$1\cdot 3^{-1} + \cdots$"), and one ready to serve the subtree for all rational numbers with 3-adic expansions beginning "$2\cdot 3^{-1} + \cdots$". As more numbers are added to the drawing, even higher levels are needed, and since there is no upper bound on p-adic distances between rational numbers, there is no upper bound on the levels in the tree for $\mathbb{Q}$. One consequence is that $\mathbb{Q}$'s tree has no root. Figure 3 depicts this tree, with selected branches and rational numbers shown. The part of this tree that represents $\mathbb{Z}$ is on the left; angles between branches in $\mathbb{Z}$'s subtree have been changed from the angles in Figure 2 in order to display the new rational numbers more easily.

As in the tree for $\mathbb{Z}$, there are leaves at the ends of only selected descending paths (although, somewhat paradoxically, at every level *above* the bottom, the tree is "full"

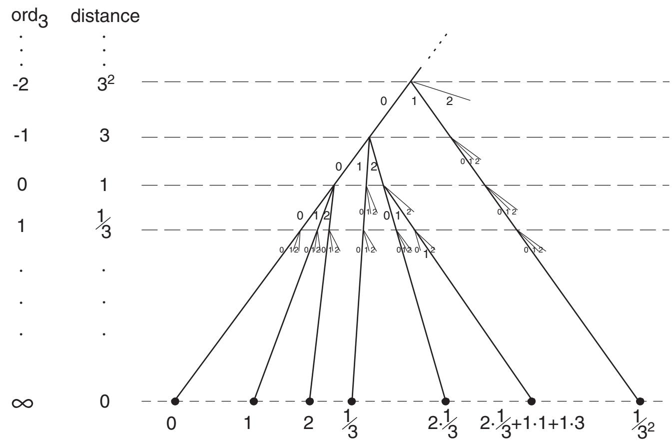

Figure 3.

in the sense that every node has branches to three nodes below it, all on the way toward leaves; this "fullness" was true of $\mathbb{Z}$'s tree as well). This leads us to the field of p-adic numbers:

Definition. The *field of p-adic numbers*, $\mathbb{Q}_p$, is the completion of $\mathbb{Q}$ under the p-adic metric, with addition and multiplication being defined in the expected way; see [**2**, pp. 53–57], [**4**, p. 10], or [**6**, p. 16].

It follows that each p-adic number has a unique *p-adic expansion* of the form $\sum_{k=n}^{\infty} b_k p^k$ for some $n \in \mathbb{Z}$ and $b_k \in \{0, 1, \ldots, p-1\}$ for each k. In addition, each such series $\sum_{k=n}^{\infty} b_k p^k$ represents a p-adic number.

Now, a tree picture for $\mathbb{Q}_p$ is just like that for $\mathbb{Q}$, but with a leaf at the end of *every* descending path, so Figure 3 serves also to display the field of 3-adic numbers.

3. TRIANGLES, DISCS, AND THE TREE PICTURE. After becoming familiar with the tree structure, an intuitive feel for distance develops easily, just as for the tree of classification of species. In addition, some of the previously surprising facts about ultrametric spaces become clear. For example, the fact that "triangles are always isoceles" is demonstrated by drawing the few different possible relative positions of three points, as in Figure 4. If two points q and r are close to one another, then their distances to a more distant point s must be the same.

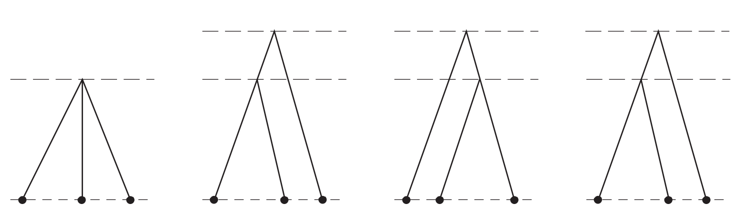

Figure 4.

The picture of a disc is also simple. Given a point q and a distance γ, the set $\{x : d(x, q) \leq \gamma\}$ is represented in an ultrametric tree by the set of all leaves in the subtree descending from a certain node—see Figure 5.

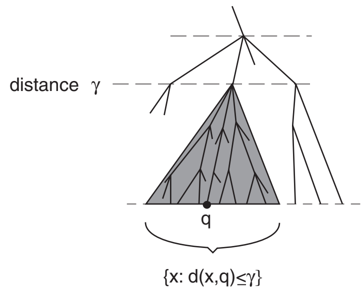

Figure 5.

With this picture, it is easy to see why every point in a given disc is actually a center of the disc: Suppose r is an arbitrary point in the disc of Figure 5. Then the disc centered at r, $\{x : d(x, r) \leq \gamma\}$, is represented by the set of leaves in the subtree descending from the (unique) node above r at level γ. But this node is the same as that above q at level γ, giving the same disc.

4. VALUED FIELDS. The tree picture can be used for valued fields other than just $\mathbb{Q}_p$.

Definition. A *valued field* is a field K with a *valuation map* $v : K^\times \to \Gamma$ from $K^\times = K \setminus \{0\}$ to a totally ordered abelian group Γ such that for all $x, y \in K^\times$:

(a) $v(x \cdot y) = v(x) + v(y)$, and

(b) $v(x + y) \geq \min\{v(x), v(y)\}$ (for $x + y \neq 0$).

The group Γ is called the *value group*. By convention, the map v is extended to all of K by setting $v(0) = \infty$, an element such that $\infty > \gamma$ and $\infty + \gamma = \gamma + \infty = \infty + \infty = \infty$ for all $\gamma \in \Gamma$. Then (a) and (b) hold for all $x, y \in K$.

The field $\mathbb{Q}_p$ is a valued field with valuation map ord_p and value group $\mathbb{Z}$. In Figure 3 the "ord_3" levels are elements of the value group; for each $x \in \mathbb{Q}_p$, the valuation $v(x)$ ($= \mathrm{ord}_p(x)$) is given by the level at which branches to x and 0 diverge. In this regard, the value group is represented by a vertical "axis", larger values toward the bottom.

This vertical value group axis is the key to developing trees for valued fields in general. Some examples of valued fields besides $\mathbb{Q}_p$ are $\mathbb{C}_p$ (the p-adic analogue of $\mathbb{C}$, i.e., the completion under the p-adic metric of the algebraic closure of $\mathbb{Q}_p$), fields of formal or convergent power series with $v(\sum c_i t^i)$ being the least i such that $c_i \neq 0$, and nonstandard extensions of $\mathbb{R}$ with the elements of positive valuation being exactly the infinitesimals. While $\mathbb{Q}_p$ has a discrete value group, other valued fields may have arbitrarily complicated ordered abelian value groups.

The goal in drawing a tree picture for a valued field is to display the same features that are displayed in the tree for $\mathbb{Q}_p$, such as the fact that every point in a given disc is a center (a disc being a set of the form $\{x : v(x - b) \geq \gamma\}$ or $\{x : v(x - b) > \gamma\}$ with b in the field and γ in the value group). To develop such a picture, one considers the value group now as labeling the levels, and places points and branches appropriately by requiring that $v(x - y)$ be the level at which branches to the two points x and y diverge. We use $\mathbb{C}_p$ to work out an example.

The valued field $\mathbb{C}_p$ has value group $\mathbb{Q}$ [**2**, p. 182], so consider a vertical axis $\mathbb{Q}$, and start a tree by putting in 0 and 1, with branches necessarily diverging at level

$v(1-0) = v(1) = 0$. We have a slight advantage here because we already know what a tree for $\mathbb{Q}_p$ looks like; we could just start with that picture and add to it. However, for purposes of demonstration, let's ignore our previous knowledge and pick a new point in $\mathbb{C}_p$, say $\sqrt{p}$, a square root of p. We know that $v(\sqrt{p}) = 1/2$, so add $\sqrt{p}$ to the picture, with a branch diverging at level 1/2 from the descending path to 0. Similarly, $\sqrt[3]{p}$, a cube root of p, can be added ($v(\sqrt[3]{p}) = 1/3$), as well as other points—see Figure 6 for the placement of selected elements in $\mathbb{C}_p$'s tree.

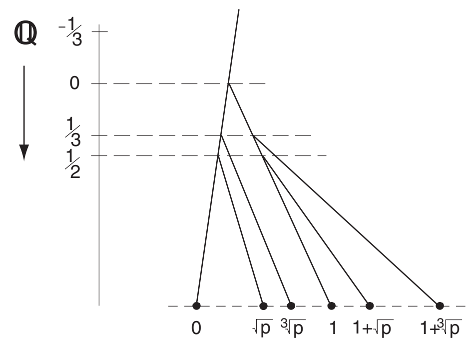

Figure 6.

While this procedure seems haphazard, it can be proved that it works [**3**]; as long as each new point q is added by extending a new branch from the descending path to the closest previously drawn point (x such that $v(x-q)$ maximal), the resulting partial tree accurately represents the set under consideration. The order in which points and branches are added is unimportant.

Although one cannot draw all the branches needed for an infinite valued field (just as one cannot mark each point of $\mathbb{Q}$ on the "number line"), the entire tree can still be visualized, with its structure displaying the structure of the valued field, discs being nice collections of points below nodes, and triangles having at least two equal sides. The valuation ring, $\{x : v(x) \geq 0\}$, also has a clear representation; this is displayed in Figure 7 for $\mathbb{C}_p$.

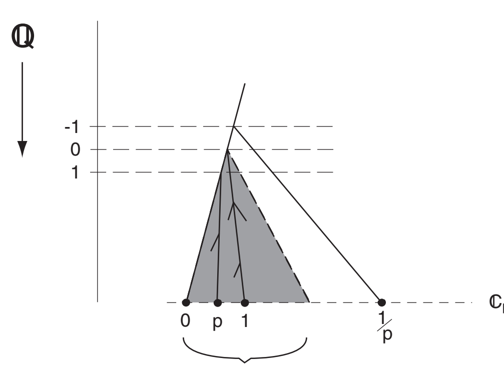

Figure 7.

5. ADDITIONAL LINES OF INVESTIGATION. This discussion is only a starting point from which more can be discovered. The reader is invited to explore, asking questions such as:

How can one see that two discs intersect only if one is contained in the other?

What does an "open" disc (of the form $\{x : v(x - b) > \gamma\}$ with b in the field and γ in the value group) look like?

What does addition look like in a p-adic tree?

What does multiplication look like?

Additional questions that may be asked by those familiar with valued fields include:

What is the form of a tree for a valued field whose value group is not a subgroup of the reals?

How is the residue field displayed in the tree of a valued field?

Where does a given linear or polynomial function send the various parts of such a tree?

How can one picture two different comparable valuations on a field by using trees?

REFERENCES

1. Albert A. Cuoco, Visualizing the p-adic integers, *Amer. Math. Monthly* **98** (1991) 355–364.
2. Fernando Q. Gouvêa, *p-adic Numbers: An Introduction*, 2nd ed., Springer-Verlag, Berlin, 1997.
3. Jan E. Holly, Canonical forms for definable subsets of algebraically closed and real closed valued fields, *J. Symbolic Logic* **60** (1995) 843–860.
4. Neal Koblitz, *p-adic Numbers, p-adic Analysis, and Zeta-Functions*, Springer-Verlag, New York, 1984.
5. R. Rammal, G. Toulouse, and M.A. Virasoro, Ultrametricity for physicists, *Rev. Modern Phys.* **58** (1986) 765–788.
6. W. H. Schikhof, *Ultrametric Calculus, an Introduction to p-adic Analysis*, Cambridge University Press, Cambridge, 1984.

JAN HOLLY is a mathematical neuroscientist who keeps returning to her roots in pure mathematics. She grew up in the Rocky Mountains, went to college at the Universities of Colorado and New Mexico, and received her Ph.D. from the University of Illinois. In the process, she discovered that someone had forgotten to install mountains in central Illinois. She proceeded to the Neurological Sciences Institute in Portland, Oregon, as a Postdoctoral Research Associate, and then to her current position as Assistant Professor at Colby College in Maine, where she can be found in the woods.

Colby College, Waterville, ME 04901

jeholly@colby.edu