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Author(s): Ben Mathes, Chris Dow and Leo Livshits
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Cantor Groups

Ben Mathes (dbmathes@colby.edu), Colby College, Waterville ME 04901; Chris Dow (cj dow@uchicago.edu), University of Chicago, Chicago IL 60637; and Leo Livshits (llivshi@colby.edu), Colby College, Waterville ME 04901

The Cantor set $C$ is simultaneously small and large. Its cardinality is the same as the cardinality of all real numbers, and every element of $C$ is a limit of other elements of $C$ (it is perfect), but $C$ is nowhere dense and it is a nullset. Being nowhere dense is a kind of topological smallness, meaning that there is no interval in which the set is dense, or equivalently, that its complement is a dense open subset of $[0, 1]$. Being a nullset is a measure theoretic concept of smallness, meaning that, for every $\epsilon > 0$, it is possible to cover the set with a sequence of intervals $I_n$ whose lengths add to a sum less than $\epsilon$.

It is possible to tweak the construction of $C$ that results in a set $\mathcal{H}$ that remains perfect (and consequently has the cardinality of the continuum), is still nowhere dense, but the measure can be made to attain any number $\alpha$ in the interval $[0, 1)$. The construction is a special case of Hewitt and Stromberg’s Cantor-like set [1, p. 70], which goes like this: let $E_0$ denote $[0, 1]$, let $E_1$ denote what is left after deleting an open interval from the center of $E_0$. If $E_{n-1}$ has been constructed, obtain $E_n$ by deleting open intervals of the same length from the center of each of the disjoint closed intervals that comprise $E_{n-1}$. The Cantor-like set is then defined to be the intersection of all the $E_n$. An example of such a construction might begin by removing the middle open subinterval of $[0, 1]$ of length $\frac{1}{3}$. You will be left with the two intervals $[0, \frac{2}{3}]$ and $[\frac{1}{3}, 1]$, from which you remove open intervals of length $\frac{1}{3}$ from the centers of both. Continuing inductively, you will have removed subintervals whose lengths add to

$$\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots,$$

resulting in a Cantor-like set with measure $\frac{1}{3}$, a set that is large in the sense of cardinality and measure, but retains its topological smallness.

Let us consider an algebraic measure of size. The unit interval is a group under addition modulo 1, and algebraically speaking, a subset of this group is small when it generates a proper subgroup of $[0, 1)$. The traditional Cantor set $C$ is algebraically large. Not only is $[0, 1)$ the smallest group containing $C$, we now show that $2C$ is already all of $[0, 1)$, where $nC$ denotes the set of all $n$-fold sums (mod 1) $\alpha_1 + \cdots + \alpha_n$ with $\alpha_i \in C$ ($1 \leq i \leq n$).

Let $w \in [0, 1)$ be given, and let $z = w/2$. Note that one can write $z$ as $z = x + y$ where the ternary (base 3) expansions of $x$ and $y$ contain only 0’s and 1’s. Indeed, if

$$z = \sum_{i=1}^{\infty} z_i \left( \frac{1}{3} \right)^i,$$

with $z_i \in \{0, 1, 2\}$, then when $z_i = 0$ we take both $x_i$ and $y_i$ equal to 0, when $z_i = 1$ we take $x_i = 1$ and $y_i = 0$, and when $z_i = 2$ we take both $x_i$ and $y_i$ equal to 1. Letting $x = \sum x_i \left( \frac{1}{3} \right)^i$ and $y = \sum y_i \left( \frac{1}{3} \right)^i$ then gives us $z = x + y$, and $w = 2x + 2y$. Thus with $c_1 = 2x$ and $c_2 = 2y$ we have two elements of $C$ for which $w = c_1 + c_2$.

We now build a Hewitt and Stromberg Cantor-like set $\mathcal{G}$ that is algebraically tiny. Let $\mu(E)$ denote the outer measure of $E$, so if $E$ is an interval, then $\mu(E)$ is just the
length of the interval. For a general set \( E \), the outer measure \( \mu(E) \) is the infimum of the set of all sums

\[
\sum_{i=1}^{\infty} \mu(I_i),
\]

for which \( I_i (i \in \mathbb{N}) \) are intervals whose union contains \( E \). The nullsets are those \( E \) with \( \mu(E) = 0 \). In the general construction of a Cantor-like set described above, upon deleting open intervals of the same length from the center of each of the disjoint closed intervals that comprise \( E_{n-1} \), one is left with \( E_n \), which is a disjoint union of exactly \( 2^n \) closed intervals, each with the same length, which we denote \( r_n \).

Note that for any two subintervals \( I \) and \( J \) of \([0, 1)\), the set

\[
I + J = \{ x + y | x \in I, y \in J \}
\]

satisfies \( \mu(I + J) \leq \mu(I) + \mu(J) \). Now \( E_n = I_1 \cup \cdots \cup I_{2^n} \) and a crude estimate gives us

\[
2E_n \subset \bigcup_{k=1}^{2^n} \bigcup_{j=1}^{2^n} I_k + I_j,
\]

so that

\[
\mu(2E_n) \leq \sum_{k=1}^{2^n} \sum_{j=1}^{2^n} \mu(I_k + I_j) \leq (2^n)^2 (2^n) r_n.
\]

Similarly we have

\[
\mu(mE_n) \leq \sum_{i_1=1}^{2^n} \cdots \sum_{i_m=1}^{2^n} \mu(I_{i_1} + \cdots + I_{i_m}) \leq (2^n)^m (m) r_n,
\]

for every natural number \( m \). We now choose \( r_n \) to force \( \lim_{n \to \infty} \mu(mE_n) = 0 \) for every \( m \); e.g., \( r_n = \frac{1}{2^n} \) is sufficient.

Let \( \mathcal{G} \) denote the Cantor-like set determined by the sequence \( r_n \), and let \( \hat{\mathcal{G}} \) denote the group generated by \( \mathcal{G} \). Thus \( \hat{\mathcal{G}} = \cap E_n \) and, since \( E_n \) contains the inverse of each of its elements, we have that \( \mathcal{G} \) contains the inverse of each of its elements. It follows that

\[
\hat{\mathcal{G}} = \{ x_1 + \cdots + x_m | m \in \mathbb{N}, x_i \in \mathcal{G}, 1 \leq i \leq m \}, \quad \text{and} \quad \hat{\mathcal{G}} \subset \bigcup_{m=0}^{\infty} \cap_{n=0}^{\infty} mE_n.
\]

Since \( \mu(mE_n) \to 0 \) for every \( m \), we see that \( \cap_{n=0}^{\infty} mE_n \) is a nullset. This shows that \( \hat{\mathcal{G}} \) is a nullset, and it also shows, for those who know about Baire category, that \( \hat{\mathcal{G}} \) is first category, since each set \( \cap_{n=0}^{\infty} mE_n \) is closed with empty interior.

**Summary.** The Cantor subset of the unit interval \([0, 1)\) is large in cardinality and also large algebraically, that is, the smallest subgroup of \([0, 1)\) generated by the Cantor set (using addition mod 1 as the group operation) is the whole of \([0, 1)\). In this paper, we show how to construct Cantor-like sets which are large in cardinality but small algebraically. In fact for the set we construct, the subgroup of \([0, 1)\) that it generates is, like the Cantor set itself, nowhere dense in \([0, 1)\).

**Reference**