Triangular Truncation of $k$-Fibonacci and $k$-Lucas Circulant Matrices

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Abstract

We prove a general theorem that gives tight bounds on the spectral norms of triangularly truncated $k$-Fibonacci and $k$-Lucas circulant matrices. The bounds are good enough to enable the calculation of the limit

$$\frac{||C||}{||\tau(C)||},$$

as the dimension $n$ approaches infinity, where $\tau(C)$ denotes the triangular truncation of $C$, and $C$ is any $n \times n$ circulant matrix built using a sequence $(s_i)$ satisfying

$$s_i = ks_{i-1} + s_{i-2}.$$

In particular, we have that this limit is equal to the golden ratio, if $C$ is built using either the ordinary Fibonacci or Lucas sequence.

1 Introduction

The $k$-Fibonacci and $k$-Lucas sequences are second order recursive sequences satisfying

$$f_{i-2} + kf_{i-1} = f_i,$$

for all integers $i \geq 2$, the two sequences determined by their initial values; the $k$-Fibonacci sequence begins with $f_0 = 0$ and $f_1 = 1$, while the $k$-Lucas sequence begins $f_0 = 2$ and $f_1 = k$. We will follow the notation used in [9] and denote the set of all such recursive sequences by $\mathcal{R}(k,1)$. When $k = 1$ we obtain the ordinary Fibonacci and Lucas sequence. There has recently

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been a flurry of interest in the norms and singular values of various matrices built using the Fibonacci and Lucas sequences, and their generalizations defined above ([1], [2], [4], [5], [8], [14], [15], [16]).

In [15], the focus is on spectral norm estimates for $r$-circulant matrices whose entries are generated via the $k$-Fibonacci or $k$-Lucas sequence for positive $k$. The $r$-circulant matrices are of the form

$$A = \begin{bmatrix}
a_0 & a_1 & \ldots & a_{n-2} & a_{n-1} \\
ar_{n-1} & a_0 & a_1 & \ldots & a_{n-2} \\
r_{n-2} & r_{n-1} & a_0 & \ldots & a_{n-3} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
r_1 & r_2 & \ldots & r_{n-1} & a_0
\end{bmatrix},$$

thus if $r = 1$, then $A$ is the $n \times n$ circulant matrix determined by the sequence $(a_i)$ (see [10] for a nice expository article on circulant matrices), while $A$ becomes the triangular truncation of a circulant matrix when $r = 0$.

Here is the main result of [15] in the case of the $k$-Fibonacci sequence:

**Theorem 1** [15] Let $A$ be as above, with $a_i = f_i$, the $k$-Fibonacci sequence.

1. If $|r| \geq 1$, then

$$\sqrt{\frac{f_{n}f_{n-1}}{k}} \leq ||A|| \leq \frac{|r| - |r|^n (f_n + |r| f_{n-1})}{1 - k|r| - |r|^2}.$$

2. If $|r| < 1$, then

$$|r| \sqrt{\frac{f_{n}f_{n-1}}{k}} \leq ||A|| \leq \frac{f_n + f_{n-1} - 1}{k}.$$

When $r = 1$, the matrix $A$ is circulant and, since its entries are non-negative,

$$||A|| = \sum_{i=0}^{n-1} f_i = \frac{f_{n-1} + f_{n-1} - 1}{k},$$

(see the preliminary section). While (2) of Theorem 1 gives no non-trivial lower bound for $r = 0$, it does say that triangular truncation of a circulant matrix whose entries are generated by a $k$-Fibonacci is contractive. The aim of this note is to give, for any non-zero $k$, tight estimates on the spectral norm of the triangularly truncated $A$. 

2
2 Preliminaries

The space $\mathcal{R}(k, 1)$ is a two dimensional (real or) complex vector space with a natural basis consisting of two sequences of the form

$$(1, \tau, \tau^2, \tau^3, \ldots),$$

corresponding to the two distinct roots $\lambda$ and $\mu$ of the polynomial $\tau^2 - k\tau - 1$. These two roots satisfy

$$0 < |\mu| < 1 < |\lambda|,$$

and we refer to $\lambda$ as the $k$-golden ratio. As it happens, $-1 < \mu < 0$ and $1 < \lambda$ for positive $k$, while $\lambda < -1$ and $0 < \mu < 1$ when $k$ is negative. One has

$$\lambda + \mu = k \text{ and } \lambda\mu = -1,$$

and writing the Fibonacci and Lucas sequences, $(f_i)$ and $(l_i)$, in terms of this basis gives the Binet formulae:

$$f_i = \frac{\lambda^i - \mu^i}{\lambda - \mu} \text{ and } l_i = \lambda^i + \mu^i,$$

for all integers $i$. Almost all of this can be found in [9].

An $n \times n$ circulant matrix corresponding to a sequence $(a_i)$ is the matrix

$$A = \begin{bmatrix}
\begin{array}{cccc}
a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \\
a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\
a_{n-2} & a_{n-1} & a_0 & \cdots & a_{n-3} \\
\vdots & \vdots & \ddots & \vdots \\
a_1 & a_2 & \cdots & a_{n-1} & a_0
\end{array}
\end{bmatrix},$$

and the set of all such circulant matrices forms a commutative algebra $C$ of normal matrices, which is simultaneously diagonalizable. If

$$E = \frac{1}{\sqrt{n}} \begin{bmatrix} u^{ij} \end{bmatrix}_{i,j=0}^{n-1},$$

with $u$ a primitive $n^{th}$-root of unity, then $E$ is the unitary that implements the Discrete Fourier Transform, and it is also the unitary that diagonalizes circulant matrices, i.e. $E^*AE$ is a diagonal matrix with diagonal entries

$$\sum_{i=0}^{n-1} a_i v^i,$$
one entry for each \(n^{th}\)-root of unity \(v\), e.g. \(v = u^i\) for \(i = 0, \ldots, n - 1\). We will refer to \(E\) as the DFT unitary. Most of this paragraph can be found in [10].

The singular values of a matrix \(A\) are the eigenvalues of \((A^*A)^{\frac{1}{2}}\), and traditionally they are listed in descending order

\[s_0 \geq s_1 \geq \cdots \geq s_m > 0,\]

with \(m + 1\) equal to the rank of \(A\). The spectral norm \(\|A\|\) is defined to be \(s_0\), and the Frobenius norm \(\|A\|_F\) is defined to be \(\sqrt{\sum_{i=0}^{m} s_i^2}\). Writing the entries \([a_{ij}]\) of the matrix \(A\), we get

\[\|A\|_F^2 = \sum_{ij} |a_{ij}|^2,\]

and

\[\|A\| = \sup_{\|x\| \leq 1} \|Ax\|,\]

where \(\|x\|\) denotes the Euclidean norm of \(x\). Thus, the spectral norm is an operator norm. When \(A\) is a diagonal matrix, then the singular values are the absolute values of the diagonal entries, and more generally, when \(A\) is normal, the singular values are the absolute values of the eigenvalues of \(A\).

In particular, if \(A\) is an \(n \times n\) circulant matrix generated from a sequence \((a_i)\), then the spectral norm of \(A\) is the supremum over

\[|\sum_{i=0}^{n-1} a_i v^i|,\]

as \(v\) varies through the \(n^{th}\) roots of unity. When \((a_i)\) is non-negative, then this supremum is attained with \(v = 1\), so that

\[\|A\| = \sum_{i=0}^{n-1} a_i.\]

In case \(F\) is the circulant corresponding to the \(k\)-Fibonacci sequence with positive \(k\), we get

\[\|F\| = \sum_{i=0}^{n-1} f_i = \frac{f_{n-1} + f_{n-2}}{k},\]

which is the upper bound of the corresponding \(r\)-circulant matrix obtained in [15], for \(|r| \leq 1\). It appears that the authors were unaware of this at the
time they wrote their paper, while for the ordinary Fibonacci sequence, this fact is explicitly noted in [8].

The circulant matrices are examples of Toeplitz operators: they are constant along each upper left to lower right diagonal. An infinite matrix \([a_{ij}]\) acting boundedly on \(\ell_2\) is an analytic Toeplitz operator if it is Toeplitz and triangular. As such, the matrix is given by a sequence \((a_i)\), and, assuming it is upper triangular, is of the form

\[
A = \begin{bmatrix}
a_0 & a_1 & a_2 & \ldots \\
0 & a_0 & a_1 & a_2 & \ldots \\
0 & 0 & a_0 & a_1 & \ldots \\
\vdots & \vdots & \vdots & \ddots & \ddots
\end{bmatrix}.
\]

The corresponding function \(f(z) = \sum_{i=0}^{\infty} a_i z^i\) is a bounded analytic function in the open unit disc, and with \(||A||\) denoting the operator norm of \(A\), one has

\[
||A|| = \sup_{|z| \leq 1} |f(z)|.
\]

If \(A\) is an \(n \times n\) truncated circulant matrix corresponding to \((a_i)\), and it is known that \(f(z) = \sum_{i=0}^{\infty} a_i z^i\) is a bounded analytic function in the unit disc, then \(A\) is a compression of \(A\), hence

\[
||A|| \leq ||A|| = \sup_{|z| \leq 1} |f(z)|.
\]

These facts can be found in any text that treats Hardy spaces: see [13], [12], and [6].

A matrix space, endowed with the spectral norm, lets us describe the distance from a matrix \(A\) to a set \(\mathcal{K}\) as

\[
\inf_{K \in \mathcal{K}} ||A - K||.
\]

If \((s_i)\) denotes the sequence of singular values of \(A\), arranged in decreasing order, then \(s_i\) is exactly the distance from \(A\) to the set of matrices of rank less than or equal to \(i\) (for \(i = 0, \ldots, \text{rank}(A))\). This fact can be found in [7]. We intend to use this fact in the following form: given \(A\), we find a matrix \(O\) of rank one with

\[
||A - O|| \leq r.
\]

We conclude that \(s_i \leq r\) for all \(i \geq 1\).
3 Results

The first thought one might have, presented with evidence that truncating particular circulant matrices is contractive, is that it might be a property of all circulant matrices rather than the particular ones. As it happens, this is not the case. In fact, the rate of growth of the truncation norm, when restricted to the set of circulant $n \times n$ matrices, is the same as the rate of growth on all $n \times n$ matrices (see [3] for a proof that the norm of the triangular truncation operator, defined on the set of all $n \times n$ matrices, grows on the order of $\log n$).

**Theorem 2** Triangular truncation is unbounded when restricted to circulant matrices. In fact, there exists $n \times n$ circulant matrices $A$ with

$$\frac{||\tau(A)||}{||A||} \geq \frac{1}{\pi} \log \left( \frac{n}{2} \right).$$

**Proof.** Let $E$ denote the DFT unitary. Let $\{e_1, \ldots, e_n\}$ denote the standard basis of $\mathbb{C}^n$. Our plan is to show that there exists an $n \times n$ diagonal matrix $D_0$ of norm one such that

$$|<E^*\tau(ED_0E^*)Ee_1, e_1>| \geq \frac{1}{\pi} \log \left( \frac{n}{2} \right).$$

Observe that for a circulant $C = \text{circ}(c_0, \ldots, c_{n-1})$, the northwest entry of the matrix $E^*\tau(C)E$ is

$$<E^*\tau(C)Ee_1, e_1> = \frac{1}{n} \sum_{i=0}^{n-1} (n - i)c_i.$$

Apply this to the circulant $C$, where $c_i = \frac{1}{n} \sum_{j=0}^{n-1} a_j u^{ij}$, and with $u$ a primitive $n^{th}$ root of unity, i.e. the circulant

$$C = EDE^*,$$

with $D = \text{diag}(a_0, \ldots a_{n-1})$, to get

$$|<E^*\tau(EDE^*)Ee_1, e_1>| = \frac{1}{n^2} \sum_{i=0}^{n-1} (n-i) \sum_{j=0}^{n-1} a_j u^{ij} = \frac{1}{n^2} \sum_{j=0}^{n-1} a_j \sum_{i=0}^{n-1} (n-i)u^{ij},$$

from which we see that the supremum over all diagonal contractions $D$ is attained, and equal to
\[ | \langle E^* \tau(ED_0 E^*)E e_1, e_1 \rangle \rangle = \frac{1}{n^2} \sum_{j=0}^{n-1} \left| \sum_{i=0}^{n-1} (n-i)u^i \right|. \]

Let’s look at a typical sum \( \sum_{i=0}^{n-1} (n-i)u^i \) with \( 1 \leq j \): let \( v \neq 1 \) denote any \( n \)th root of unity. We massage the sum to appear like the derivative of a finite geometric series, obtaining

\[ \left| \sum_{i=0}^{n-1} (n-i)v^i \right| = |(n+1) \sum_{i=0}^{n-1} v^i - \sum_{i=0}^{n-1} (i+1)v^i| = |\sum_{i=0}^{n-1} (i+1)v^i| = \frac{n}{|1-v|}. \]

When \( v = e^{2\pi ki/n} \), we have

\[ |1-v| \leq \frac{2k\pi}{n}, \]

and realizing the distances \( |1-v| \) come in conjugate pairs, we deduce

\[ | \langle E^* \tau(ED_0 E^*)E e_1, e_1 \rangle \rangle \geq \frac{1}{n^2} \left( \frac{n(n+1)}{2} + 2 \sum_{k=1}^{n-1} \frac{n}{|1-e^{2\pi ki/n}|} \right) \]
\[ \geq \frac{1}{n^2} \left( \frac{n(n+1)}{2} + 2 \sum_{k=1}^{n-1} \frac{n^2}{2k\pi} \right) \]
\[ \geq \frac{1}{\pi} \sum_{k=1}^{n-1} \frac{1}{k} \geq \frac{1}{\pi} \log \left( \frac{n^2}{\pi} \right) \]

\[\square\]

We let \( \Lambda \) denote the \( n \times n \) circulant matrix generated by the sequence \( (\lambda^i) \), and \( M \) the \( n \times n \) circulant matrix generated by \( (\mu^i) \), (with \( \lambda \) and \( \mu \) the roots of \( \tau^2 - k\tau - 1 \) as above, and \( \lambda \) the \( k \)-golden ratio). The general \( n \times n \) circulant generated from a sequence in \( R(k, 1) \) is then

\[ C = \alpha \Lambda + \beta M, \]

with \( \alpha, \beta \in \mathbb{C} \). Its triangular truncation we denote by

\[ T = \tau(C) = \alpha \tau(\Lambda) + \beta \tau(M). \]

Assume from now on that \( k \) is any non-zero real number.

**Lemma 3** Let \( (s_i) \) denote the sequence of singular values of \( \tau(\Lambda) \). Then

\[ s_i \leq \frac{1}{|\lambda| - 1} \]

for all \( i \geq 1 \).
Proof. Let $O$ denote the rank one $n \times n$ matrix

$$O = \begin{bmatrix}
1 \\
\lambda^{-1} \\
\lambda^{-2} \\
\vdots \\
\lambda^{-n+1}
\end{bmatrix} [1 \; \lambda \; \lambda^2 \ldots \lambda^{n-1}].$$

We then have that $O - \tau(\Lambda)$ is a strictly lower triangular Toeplitz matrix corresponding to the bounded analytic function

$$f(z) = \sum_{i=1}^{\infty} \lambda^{-i} z^i = \frac{z}{\lambda - z},$$

so that

$$||O - \tau(\Lambda)|| \leq \sup_{|z| < 1} \left| \frac{z}{\lambda - z} \right| = \frac{1}{|\lambda| - 1}.$$  \hfill \Box

**Lemma 4** With $(s_i)$ still denoting the sequence of singular values of $\tau(\Lambda)$, we have

$$||\tau(\Lambda)||^2_F = \sum_{i=0}^{n-1} s_i^2 = \frac{n + \lambda^2(\lambda^{2n} - n - 1)}{(1 - \lambda^2)^2}$$

and

$$\sqrt{n + \lambda^2(\lambda^{2n} - n - 1) - (n-1)(1-|\lambda|)^2} \leq ||\tau(\Lambda)|| \leq \sqrt{n + \lambda^2(\lambda^{2n} - n - 1) - (n-1)(1-|\lambda|)^2}.$$  \hfill \Box

Proof. We compute

$$||\tau(\Lambda)||^2_F = \sum_{t=1}^{n} \sum_{i=0}^{t-1} (\lambda^i)^2 = \sum_{t=1}^{n} \frac{1 - \lambda^{2t}}{1 - \lambda^2} = \frac{n + \lambda^2(\lambda^{2n} - n - 1)}{(1 - \lambda^2)^2}. $$

Also, we note that

$$||\tau(\Lambda)|| = \sqrt{\sum_{i=0}^{n-1} s_i^2 - \sum_{i=1}^{n-1} s_i^2} \geq \sqrt{n + \lambda^2(\lambda^{2n} - n - 1) - (n-1)(1-|\lambda|)^2},$$

by Lemma 3. The other inequality follows from $||\tau(\Lambda)|| \leq ||\tau(\Lambda)||_\infty$.  \hfill \Box
Example 1  The bounds in the previous lemma are fairly tight. For exam-
ple, with \( n = 20 \) and \( k = 1 \), a mathematica computation gives

\[
15126.9975 \leq 15126.9991 \leq 15126.9992
\]

for the spectral norm and its upper and lower bounds. For \( n \geq 3 \), the
difference between the upper and lower bounds is decreasing to zero, so the
estimates tighten as \( n \) increases.

Lemma 5  We have

\[
||\tau(M)|| \leq \frac{\lambda}{|\lambda| - 1},
\]

\[
||M|| = \frac{1 - |\mu|^n}{1 - |\mu|} = \frac{|\lambda|^n - 1}{|\lambda|^n - 1 (|\lambda| - 1)} = \frac{|\lambda| - \frac{1}{|\lambda|^{n-1}}}{|\lambda| - 1},
\]

and

\[
||\Lambda|| = \frac{1 - |\lambda|^n}{1 - |\lambda|}.
\]

Proof.  The function \( f(z) = \sum_{i=0}^{\infty} \mu^i z^i = \frac{1}{1 - \mu z} \) has

\[
\sup_{|z|<1} |f(z)| = \frac{1}{1 - |\mu|} = \frac{|\lambda|}{|\lambda| - 1},
\]

which proves the first inequality. The circulant matrix \( M \) has spectral norm
\( \sum_{i=0}^{n-1} \mu^i \) for positive \( \mu \), and \( |\sum_{i=0}^{n-1} \mu^i v^i| \) is maximized when \( v = -1 \) for
negative \( \mu \): in both cases the norm is \( \sum_{i=0}^{n-1} |\mu|^i = \frac{1 - |\mu|^n}{1 - |\mu|} \). The proof for \( \Lambda \)
is similar.

\[\square\]

Theorem 6  Assume that \( C \) and \( T \) are defined as above, with \( \alpha \neq 0 \). We
have

\[
\frac{||C||}{||T||} \to \frac{1 + |\lambda|}{|\lambda|},
\]

as \( n \to \infty \).
Proof. Using the triangle inequality, and the inequality
\[ |\alpha||A|| - |\beta||B| \leq ||\alpha A + \beta B||, \]
in both the numerator and the denominator, we use the previous lemmas to arrive at
\[
\frac{\alpha |1-|\lambda|^n| - |\beta| |\lambda|-\frac{1}{|\lambda|-1}}{\alpha \sqrt{\frac{\lambda^2(\lambda^{2n}-n+1)}{(1-\lambda^2)^2}} + |\beta| |\lambda|-\frac{1}{|\lambda|-1}} \leq \frac{||C||}{||T||} \leq \frac{\alpha |1-|\lambda|^n| + |\beta| |\lambda|-\frac{1}{|\lambda|-1}}{\alpha \sqrt{\frac{\lambda^2(\lambda^{2n}-n+1)}{(1-\lambda^2)^2}} - \frac{n-1}{(1-|\lambda|)^2} - |\beta| |\lambda|-\frac{1}{|\lambda|-1}}.
\]
Upon taking the limit as \( n \to \infty \), we have the theorem.

\[ \square \]

**Example 2** If \( C \) is built using the ordinary Fibonacci or Lucas sequence, and \( T \) is the truncated matrix (as in the previous theorem), then
\[
\frac{||C||}{||T||} \to \text{golden ratio},
\]
since in this case \( \lambda = \frac{\lambda + 1}{\lambda} \).

**Example 3** Another famous integer sequence is the Pell sequence
\[ 0, 1, 2, 5, 12, \ldots \]
which is the element of \( \mathcal{R}(2,1) \) beginning with 0 and 1. When the circulant matrix is built using the Pell sequence, we have
\[
\frac{||C||}{||T||} \to \sqrt{2}.
\]

4 Estimating the norm of a Fibonacci Matrix

In this section we demonstrate a technique for estimating the norm of a truncated Fibonacci circulant matrix. We demonstrate the technique with the matrix \( T = \tau(C) \), with \( C \) the circulant matrix built with the Fibonacci sequence
\[ 1, 1, 2, 3, \ldots \]
Our technique was inspired by the methods used in [11], and in particular, the lower bound is obtained as a consequence of Theorem 3.4 in [11]. While this theorem is true, the proof in [11] has a serious flaw: it is based upon their Lemma 3.3, which is false. We fix this gap by proving a stronger result.

**Theorem 7** If \( s_0, s_1, \ldots, s_{n-1} \) are the singular values of the \( n \times n \) matrix \( T \), then \( s_i \leq 1 \) for all \( i \in \{1, \ldots, n-1\} \).

Our proof of this theorem requires, in addition to well known Fibonacci identities (which can be found in [17]), the following identities that are hard to find in the literature. Their proofs are easy induction exercises.

**Lemma 8** For all integers \( m, k, \) and \( t \), we have

\[
f_m f_k - f_{m-t} f_{k+t} = (-1)^{k+1} f_{m-k-t} f_t.
\]

**Lemma 9** For all integers \( n \) and \( 1 \leq i < j \leq n \) with \( j - i \) odd, we have

\[
\sum_{k=i}^{j-1} (-1)^{k-i+1} f_{n-k} f_k = f_{j-1} f_{n-j} - f_i f_{n-i+1}.
\]

**Lemma 10** For all integers \( n \) and \( 1 \leq i \leq j \leq n \) with \( 2 \leq j - i \) even, we have

\[
\sum_{k=i}^{j-1} (-1)^{k-i+1} f_{n-k} f_k = f_n - f_{j-1} f_{n-j} - f_i f_{n-i+1}.
\]

**Proof.** (of Theorem 7) Assume that \( O \) is the rank one matrix determined by the property that \( T - O \) has all zeros in the first row and the last column. Specifically, we write

\[
O = [a_{ij}] = \begin{bmatrix}
\frac{f_1}{f_n} & \frac{f_2}{f_n} & \cdots & \frac{f_n}{f_n}
\end{bmatrix}
\begin{bmatrix}
f_n \\
f_{n-1} \\
r \vdots \\
f_1
\end{bmatrix}
\]

and thus

\[
a_{ij} = \frac{f_{n-i+1} f_j}{f_n}
\]

for \( i, j = 1, \ldots n \).
Our strategy is to prove that \( \|T - O\| \leq 1 \), which we do by proving \( \|AA^*\| \leq f_n^2 \), when \( A \equiv f_n(T - O) \). Notice that, with \( A = [a_{ij}] \), we have

\[
a_{ij} = \begin{cases} 
  f_n f_{j-i+1} - f_j f_{n-i+1} & i \leq j \\
  -f_j f_{n-i+1} & i > j
\end{cases}
\]

as a consequence of Lemma 8. Let \( v_i \) denote row \( i \) of the matrix \( A \), and compute the \( i, j \)-entry of \( AA^* \), which is \( <v_i, v_j> \). With no loss of generality, assume that \( i \leq j \). When \( j - i \) is odd, we get

\[
<v_i, v_j> = \sum_{k=i}^{j-1} f_k^2 f_{n-i+1} f_{n-j+1} + \sum_{k=i}^{j-1} (-1)^{k-i+1} f_{n-k} f_{i-1} f_k f_{n-j+1} - \sum_{k=j} f_k^2 f_{n-k} f_{i-1} f_{j-1}.
\]

We use the well known identity

\[
\sum_{k=1}^{s} f_k^2 = f_s f_{s+1}
\]

on the first and last summands, and use Lemma 9 on the middle summand, obtaining

\[
<v_i, v_j> = f_{i-1} f_i f_{n-i+1} f_{n-j+1} + f_{i-1} f_{n-j+1} (f_{j-1} f_{n-j} - f_i f_{n-i+1}) - f_{i-1} f_{j-1} f_{n-j} f_{n-j+1} = 0.
\]

When \( 0 \leq j - i \) is even, we use Lemma 10 obtaining

\[
<v_i, v_j> = \sum_{k=i}^{j-1} f_k^2 f_{n-i+1} f_{n-j+1} + \sum_{k=i}^{j-1} (-1)^{k-i+1} f_{n-k} f_{i-1} f_k f_{n-j+1} + \sum_{k=j} f_k^2 f_{n-k} f_{i-1} f_{j-1} = f_n f_{i-1} f_{n-i+1} f_{n-j+1} + f_{i-1} f_{j-1} f_{n-j} f_{n-j+1} + f_{i-1} f_{j-1} f_{n-j} f_{n-j+1} = f_n f_{i-1} f_{n-i+1} f_{n-j+1}.
\]

It follows that, with \( i \leq j \), the symmetric matrix \( AA^* \) is determined by the values

\[
<v_i, v_j> = \begin{cases} 
  f_n f_{i-1} f_{n-j+1} & \text{if } j - i \text{ even} \\
  0 & \text{if } j - i \text{ odd}
\end{cases}
\]
Life is simplified a little if we delete the row and column of zeros, and let $w_i$ denote row $i$ of the resulting $(n - 1) \times (n - 1)$ matrix. Now we have

$$< w_i, w_j > = \begin{cases} f_n f_i f_{n-j} & \text{if } j - i \text{ even} \\ 0 & \text{if } j - i \text{ odd} \end{cases}$$

Being both nonnegative and symmetric, we have that the largest row sum dominates the norm (see [7]). Using the well known identities

$$\sum_{i=0}^{k-1} f_{2i+1} = f_{2k}$$
$$\sum_{i=0}^{k} f_{2i} = f_{2k+1} - 1,$$

we see that each row sum is of the form

$$f_n f_i X + f_n f_{n-i} Y,$$

where $X$ is either $f_{n-i+1}$ or $f_{n-i+1} - 1$ and $Y$ is either $f_{i-1}$ or $f_{i-1} - 1$ (depending on whether $i$ and $n - i$ are even or odd). In every case, the row sums are dominated by

$$f_n f_i f_{n-i+1} + f_n f_{n-i} f_{i-1} = f_n^2,$$

which completes the proof.

□

**Example 4** The well known identity

$$\sum_{i=0}^{n-1} f_i f_{i+1} = \begin{cases} f_n^2 & \text{if } n \text{ even} \\ f_n^2 - 1 & \text{if } n \text{ odd} \end{cases}$$

lets one easily calculate the Frobenius norm of $T$, it is

$$||T||_F = \begin{cases} f_{n+1} & \text{if } n \text{ odd} \\ \sqrt{f_n^2 + 1} & \text{if } n \text{ even} \end{cases}.$$  

The fact that the spectral norm of $T$ grows rapidly, while all other singular values remain bounded by 1, explains why the Frobenius norm, always an upper bound of the spectral norm, quickly becomes a good approximation of the spectral norm for $T$. The previous theorem gives a lower bound. Indeed, since

$$s_n^2 + (n - 1) \geq \text{tr} (TT^*) = f_{n+1}^2 - \delta_n,$$

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with $\delta_n$ either zero or one, we have
\[ \sqrt{f_{n+1}^2 - n} \leq ||T|| \leq f_{n+1}. \]
When $n = 12$, mathematica gives the three values
\[ 232.974 \leq 232.988 \leq 233, \]
which is already quite tight. The corresponding circulant matrix $F$ (notation as in the beginning of section 4) has norm
\[ ||F|| = f_{n+2} - 1 \]
(recall from the preliminary section that the norm equals the sum $\sum_{i=0}^{n} f_i$), and for $n = 12$ we have the three ratios $\frac{||F||}{f_{n+1}} \leq \frac{||F||}{||T||} \leq \frac{||F||}{\sqrt{f_{n+1}^2 - n + 1}}$, computed in mathematica, as
\[ 1.6137339 \leq 1.6138174 \leq 1.6138974. \]
By the time $n$ gets to 24, all three values are within 5 decimal places of the golden ratio.

References


