MA121, Spring 2008 — Problem Set 6 Solutions

1. Problems from the textbook:

   a. Section 5.2: 20, 26, 27.

   20. Boring but straightforward.

   26. The graph looks like two triangles with equal areas, one above and one below the axis, so the integral is zero. We checked this in class by actually computing it. The Riemann sum should come close to zero, but not exactly zero.

   27. Should have been really easy: (a) is 13, (b) is -2, (c) is 11, (d) is 15.

   b. Section 5.3: 2, 4, 16, 20.

   2. It is meters/second.

   4. The integral represents the overall change of position between $t = 1$ and $t = 4$.

   16. Mathematics is so beautiful; it always frustrates me when it turns into applied stuff. For (a), it's just a matter of the definition of the integral: the total consumption is

$$\int_0^5 f(t)\,dt.$$

   For (b), we're actually given a formula, $f(t) = 32e^{0.05t}$. Now that we know how to compute integrals, we can find the exact value:

$$\int_0^5 32e^{0.05t}\,dt = \begin{bmatrix} u = 0.05t \\ du = 0.05dt \end{bmatrix} = 32\int_0^{0.25} \frac{1}{0.05}e^u\,du$$

$$= 32 \cdot 20\,[e^u]_0^{0.25} = 640(e^{0.25} - 1) \approx 181.8$$

   The problem actually tells you to compute a left-hand sum, with for this function is the same as a lower sum, so you'll get something a bit less than this. For part (c), each of the terms represents the approximate consumption during one year.

   20. The number we want is $\frac{5}{10}\int_0^{30} f(t)\,dt$, where $f(t)$ is the function in the picture. The area under the line is $600,000$, so the company will have to pay $300,000.

c. Section 5.4: 17, 18, 26, 28.

17. We get $12 - 8^2 = 12 - 64 = -52$.

18. We get $2c_1 + 12c_2^2$.

26. Both are zero, since the functions are odd and the integrals go from $-a$ to a.

28. The maximum value of $\sqrt{1+x^3}$ in the interval from 0 to 2 happens when $x = 2$, and the minimum when $x = 0$. They are 1 and 3 respectively. Since the length of the interval of integration is 2, the inequality follows at once.

2. Suppose you know that

$$\int_a^b f(x)\,dx = 18, \qquad \int_a^b g(x)\,dx = 5, \qquad \text{and} \qquad \int_a^b h(x)\,dx = -11.$$

Evaluate as many of the following as you can by using the properties of integrals. (Some of them may not be possible to determine with the data you have been given!)

a. $\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx = 18 + 5 = 23.$

b. $\int_a^b (f(x) - g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = 18 - 5 = 13.$

c. $\int_a^b f(x)g(x)\,dx$ cannot be determined.

d. $\int_a^b (g(x) + h(x))\,dx = \int_a^b g(x)\,dx + \int_a^b h(x)\,dx = 5 + (-11) = -6.$

e. $\int_a^b \frac{g(x)}{h(x)}dx$ cannot be determined.

f. $\int_a^b (f(x)+g(x)+h(x))\,dx = \int_a^b f(x)\,dx+\int_a^b g(x)\,dx+\int_a^b h(x)\,dx = 18+5-11 = 12.$

3. Given that

$$\int_0^1 f(x)\,dx = \frac{4}{3}, \qquad \int_1^2 f(x)\,dx = \frac{8}{3}, \qquad \text{and} \qquad \int_0^3 f(x)\,dx = \frac{11}{3},$$

find the values of

$$\text{a.} \quad \int_0^2 f(x)\,dx \qquad \text{b.} \quad \int_1^3 f(x)\,dx \qquad \text{c.} \quad \int_2^3 f(x)\,dx$$

a. $\displaystyle\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx = \frac{4}{3} + \frac{8}{3} = 4.$

b. $\displaystyle\int_1^3 f(x)\,dx = \int_0^3 f(x)\,dx - \int_0^1 f(x)\,dx = \frac{11}{3} - \frac{4}{3} = \frac{7}{3}.$

c. $\displaystyle\int_2^3 f(x)\,dx = \int_0^3 f(x)\,dx - \int_0^2 f(x)\,dx = \frac{11}{3} - 4 = -\frac{1}{3}.$

4. What is $\displaystyle\int_{-3}^{3} (x+5)\sqrt{9-x^2}\,dx$ equal to?

(Hints: First, use linearity to write this as the sum of two integrals. Second, remember that the integral can be interpreted as an area under a curve; what do the graphs of $y = x\sqrt{9-x^2}$ and $y = \sqrt{9-x^2}$ look like?)

Well,

$$\int_{-3}^{3} (x+5)\sqrt{9-x^2}\,dx = \int_{-3}^{3} x\sqrt{9-x^2}\,dx + \int_{-3}^{3} 5\sqrt{9-x^2}\,dx.$$

For the first term, notice that $f(x) = x\sqrt{9-x^2}$ is an odd function, and therefore the integral is zero.

For the second term,

$$\int_{-3}^{3} 5\sqrt{9-x^2} = 5\int_{-3}^{3} \sqrt{9-x^2} = 5 \times \text{area of a semicircle of radius } 3 = \frac{45\pi}{2}.$$

So the overall answer is $45\pi/2$.

5. Four calculus students disagree as to the value of the integral

$$\int_0^{\pi} \sin^8(x)\,dx.$$

Jack says that it is equal to π, Joan says that it is equal to $\frac{35\pi}{128}$. Ed claims it is equal to $\frac{3\pi}{90} - 1$, while Lesley says it is equal to $\frac{\pi}{2}$. One of them is right; which one is it?

(Hint: do *not* try to evaluate the integral! Instead, try to eliminate the three wrong answers.)

Ed is clearly wrong, since his answer is negative and $\sin^8(x)$ is always positive. The integral from 0 to π of 1 is equal to π, and $\sin^8(x)$ is usually quite a bit less than 1. So Jack is wrong too. The toughest one to eliminate is Lesley. Here are things I tried that didn't quite do it:

- Divide into four subintervals and get an upper sum. No good, because the upper sum comes out bigger than $\pi/2$.
- Use the inequality $\sin^8(x) \leqslant \sin(x)$, and so the integral is smaller than the integral of the sine. But that comes out to be 2, which is bigger than $\pi/2$.

So this is a pain.

There are, of course, several ways to get a better estimate. Perhaps the easiest is to use a six-step subdivision and compute an upper sum: divide the interval at 0, $\pi/6$, $\pi/3$, $\pi/2$, $2\pi/3$, $5\pi/6$, π. The largest values of $\sin^8(x)$ occur closest to $\pi/2$, so for those intervals they are:

- $(\sin(\pi/6))^8 = (1/2)^8 = 1/256$
- $(\sin(\pi/3))^8 = (\sqrt{3}/2)^8 = 81/256$
- $(\sin(\pi/2))^8 = 1$
- the last three intervals have the same maximum values

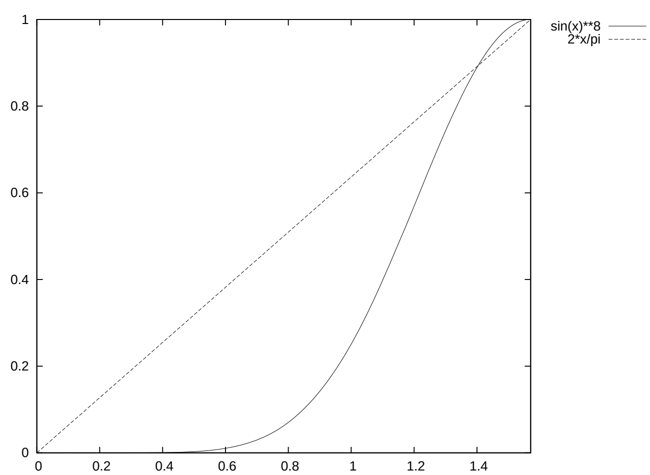
All the subintervals that length $\pi/6$, so the upper sum is

$$2\left(\frac{1}{256}\frac{\pi}{6} + \frac{81}{256}\frac{\pi}{6} + 1\frac{\pi}{6}\right) = \frac{169\pi}{384} < \frac{\pi}{2},$$

which eliminates Lesley's answer.

That means Joan must be right.

Here's an inexact but rough-and-ready way to convince oneself that Lesley's answer is wrong: it seems at first glance that the graph of $y = \sin^8(x)$ is always beneath the the straight line connecting $(0,0)$ to $(\pi/2, 1)$, i.e., beneath the graph of $y = 2x/\pi$. This is actually *not* true, as the graph shows:



But it's a close thing. So compare half our integral with the area of the triangle. The triangle has base $\pi/2$ and height 1, so it has area $\pi/4$. Doubling it gives $\pi/2$. Since the area we're computing is almost always quite a bit inside the triangle, it's probably safe to assume it's less than $\pi/2$.

This is one of those problems that loses its charm in the presence of calculators that can integrate. . .

6. Use the geometric interpretation of the integral as an area to explain why it is true that

$$\int_0^1 x^n \, dx + \int_0^1 x^{1/n} \, dx = 1$$

for any positive integer n.

Draw the square determined by $0 \leqslant x \leqslant 1$ and $0 \leqslant y \leqslant 1$. The total area of the square is 1 and the graph of $y = x^n$ divides the square into two parts. The sum of those areas is the area of the square, which is 1.

It's easy to see that the area under the curve $y = x^n$ and inside the square is exactly the first integral.

For the remaining area, look at the world "from the y-axis." We have a function of y, namely $x = y^{1/n}$ giving the height of a curve over the base interval that goes from $y = 0$ to $y = 1$. So the area is

$$\int_0^1 y^{1/n} \, dy,$$

which of course is the second integral.