MA121, Spring 2008 — Problem Set 7 Solutions

1. Problems from the textbook:

   a. Section 6,2, problems 44, 48, 54, 58, 66, 68.

   I think adding $+C$ at the end of each of the antiderivatives is silly, so I won't put it there. Of course, you can always add a constant to any antiderivative.

44. $\int (x^3 - 2)\, dx = \frac{x^4}{4} - 2x$

48. $\int \frac{4}{t^2}\, dt = -\frac{4}{t}$

54. $\int (\pi + x^{11})\, dx = \pi x + \frac{x^{12}}{12}$

58. $\int \frac{1}{e^z}\, dz = \int e^{-z}\, dz = -e^{-z}$

66.

$$\begin{aligned}\int_1^2 \frac{1+y^2}{y}\, dy &= \int_1^2 (y^{-1} + y)\, dy \\ &= \left[\ln(y) + \frac{y^2}{2}\right]_1^2 \\ &= \ln(2) + 4 - \frac{1}{2} = \frac{7}{2} + \ln(2)\end{aligned}$$

68.

$$\begin{aligned}\int_0^{\pi/4} (\sin(t) + \cos(t))\, dt &= [-\cos(t) + \sin(t)]_0^{\pi/4} \\ &= -\cos(\pi/4) + \sin(\pi/4) + \cos(0) - \sin(0) \\ &= 1\end{aligned}$$

   b. Section 6.4, problems 18, 21, 24, 33–36.

18. Just flip the endpoints, then use the Fundamental Theorem:

$$\frac{d}{dt}\int_t^\pi \cos(z^3)\, dz = \frac{d}{dt}\left[-\int_\pi^t \cos(z^3)\, dz\right] = -\cos(t^3).$$

21. OK, so $g(x)$ is the antiderivative of $f(x)$ gotten by integrating from 0 to x.

(a) $g(0)$ is the integral from 0 to 0, so $g(0) = 0$.

(b) $g'(x) = f(x)$ by the Fundamental Theorem, so $g'(1) = f(1) = -2$ (or close to that).

(c) g is concave up when g'' is positive. Now $g' = f$, so $g'' = f'$, so g'' is positive when f' is positive, hence when f is increasing. Looking at the graph, we see that this happens roughly between $x = 1$ and $x = 6$.

(d) The maximum value of $g(x)$ must happen either at one of the endpoints or at a critical point, i.e., at a value of x for which $g'(x) = 0$. Since $g' = f$, the critical points are the ones where $f(x) = 0$, i.e., they are $x = 0$ and $x = 3$. So the only candidates for a maximum of $g(x)$ are $x = 0, 3, 8$.

Now, at a maximum we expect the function to first increase, then decrease, so we expect the derivative to switch from positive to negative. That's *not* what happens at $x = 3$; in fact, $x = 3$ is a minimum point of $g(x)$. Also worth noting is that $g(3)$ is *negative*, so smaller than $g(0)$ anyway.

Since $g(0) = 0$ and the area of the bit between 3 and 8 looks bigger than the area between 0 and 3 (which will count negatively), it seems that $g(8)$ is positive. So it's the largest of the three candidate values $g(0)$, $g(3)$, $g(8)$. So the value of x that makes $g(x)$ maximum is $x = 8$.

24. Chain rule, chain rule! (Plus, discussed to death in class.)

$$\frac{d}{dx}\int_2^{x^3} \sin(t^2)\,dt = 3x^2 \sin(x^6)$$

33–36. The point here is to have you handle a function that is defined as an integral and for which there is no easier formula. It's helpful to notice that the Fundamental Theorem says that

$$\mathrm{erf}'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}.$$

So here we go:

33 Just the product rule:

$$\frac{d}{dx}(x\,\mathrm{erf}(x)) = 1\,\mathrm{erf}(x) + x\,\mathrm{erf}'(x) = \mathrm{erf}(x) + \frac{2}{\sqrt{\pi}} e^{-x^2}$$

34 Now the chain rule:

$$\frac{d}{dx}(\mathrm{erf}(\sqrt{x})) = \mathrm{erf}'(\sqrt{x})\frac{1}{2\sqrt{x}} = \frac{2}{\sqrt{\pi}}e^{-x}\frac{1}{2\sqrt{x}} = \frac{e^{-x}}{\sqrt{\pi x}}$$

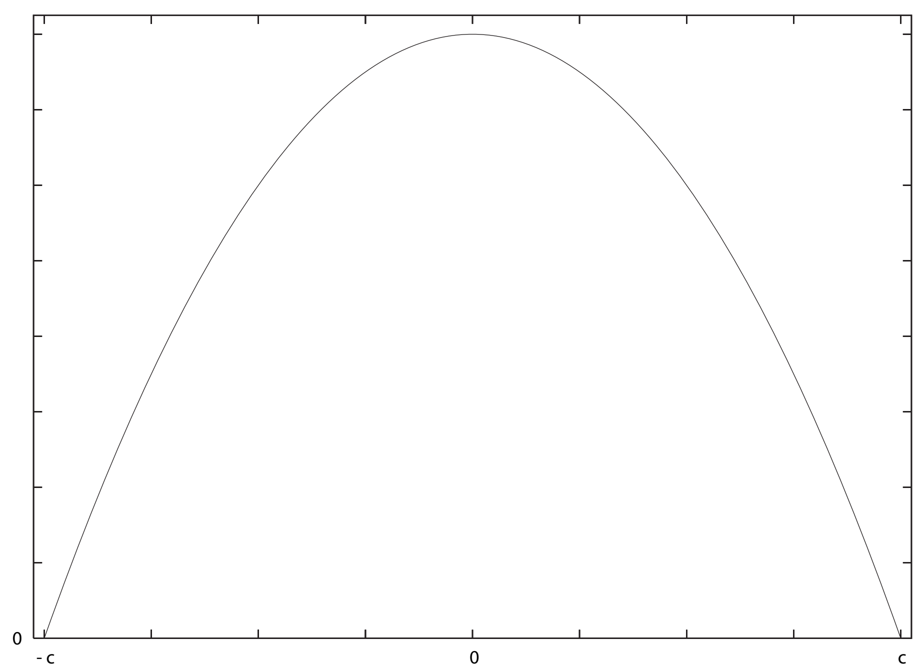
35 These seem to have little to do with erf... Just use the Fundamental theorem plus chain rule:

$$\frac{d}{dx}\int_0^{x^3} e^{-t^2}\,dt = 3x^2e^{-x^6}$$

36 Fundamental theorem plus chain rule plus additivity of the integral:

$$\frac{d}{dx}\int_x^{x^3} e^{-t^2}\,dt = \frac{d}{dx}\left[\int_0^{x^3} e^{-t^2}\,dt - \int_0^{x} e^{-t^2}\,dt\right] = 3x^2e^{-x^6} - e^{-x^2}$$

2. The area between the graph of the function $f(x) = c^2 - x^2$ and the x-axis is 36. Find the value of c.



What is given is that

$$\int_{-c}^{c}(c^2 - x^2)\,dx = 36.$$

Now, c^2 is a constant, so its integral is just the product of the constant by the length of the integral:

$$\int_{-c}^{c} c^2\,dx = c^2(2c) = 2c^3.$$

The other part is standard:

$$\int_{-c}^{c} x^2\,dx = \left[\frac{x^3}{3}\right]_{-c}^{c} = \frac{2c^3}{3}.$$

So, overall,

$$\int_{-c}^{c} (c^2 - x^2)\,dx = 2c^3 - \frac{2c^3}{3} = \frac{4c^3}{3}.$$

So we need to solve

$$\frac{4c^3}{3} = 36,$$

which boils down to $c = 3$.

3. Let R be the region under the graph of the function $f(x) = (x-2)^2$ for $-1 \leqslant x \leqslant 1$. Find a number c such that the line $x = c$ splits the region in a $2:1$ ratio.

We want to choose a number c so that

$$\int_{-1}^{c} (x-2)^2\,dx = 2\int_{c}^{1} (x-2)^2\,dx,$$

or, equivalently, so that

$$\int_{-1}^{c} (x-2)^2\,dx = \frac{2}{3}\int_{-1}^{1} (x-2)^2\,dx.$$

Computing the integrals, we get

$$\frac{(c-2)^3}{3} + 9 = \frac{2}{3}\left(\frac{-1}{3} + 9\right) = \frac{52}{9},$$

$$(c-2)^3 = 3\left(\frac{52}{9} - 9\right) = \frac{52}{3} - 27 = \frac{-29}{3},$$

which means $c = 2 - \sqrt[3]{29/3}$. What a horrible answer!

4. Let R_a the region enclosed by the graph of the parabola

$$y = \frac{2}{a^2}x - \frac{1}{a^3}x^2$$

(for some constant $a > 0$) and the x-axis. Show that the area of R_a does not depend on a. How large is the area? What curve is determined by the vertices of all these parabolas?

The easiest way to show the area of R_a is independent of a is to compute it. The parabola intersects the x-axis when $y = 0$. Factoring to find the other root, we see that the other instersection is at $x = 2a$. So the area is

$$\int_0^{2a} \left(\frac{2}{a^2}x - \frac{1}{a^3}x^2\right)\, dx = \left[\frac{x^2}{a^2} - \frac{x^3}{3a^x}\right]_0^{2a} = \frac{(2a)^2}{a^2} - \frac{(2a)^3}{3a^3} = 4 - \frac{8}{3} = \frac{4}{3}.$$

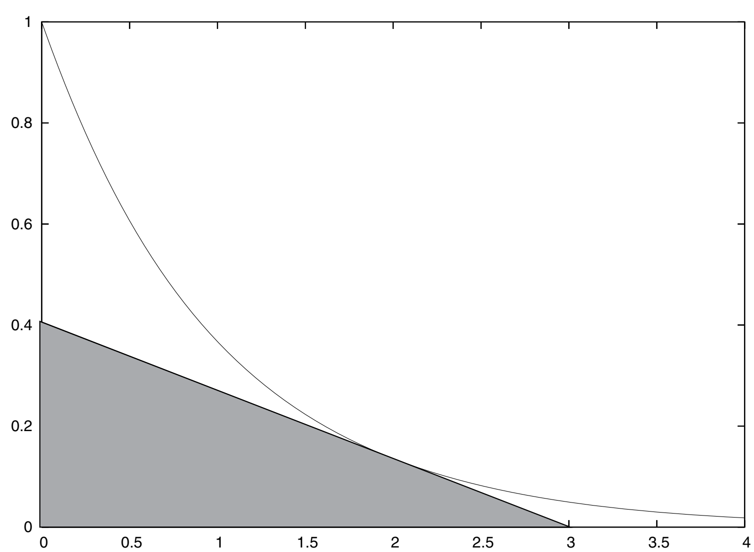
The vertex of the parabola is at its maximum point. Since $y' = \frac{2}{a^2} - \frac{2}{a^3}x$, we have $y' = 0$ when $x = a$, i.e., midway between the two roots. (Just knowing that the vertex is always between the two roots is an acceptable way to find it.)

Plugging $x = a$ into the equation, we find that the vertex is at the point $(a, \frac{1}{a})$, so the curve determined by all the vertices is just the hyperbola given by $y = 1/x$.

Notice that this means that the region R_a is the part of the parabola enclosed by a rectangle of base $2a$ and height $\frac{1}{a}$. The area of this rectangle is 2 (independent of a) and the parabola is always $\frac{2}{3}$ of the enclosing parallellogram (by an old theorem of Archimedes!). So we see another reason for the area not to depend on a.

5. (This one is quite hard!) What is the area of the largest triangle that can be formed in the first quadrant by the x-axis, the y-axis, and a line tangent to the graph of $y = e^{-x}$?

(Hints: The figure below shows one such triangle, using the tangent line through $(2, e^{-2})$. You need to find the value a so that taking the tangent line at (a, e^{-a}) gives the largest triangle. So the problem has two steps: compute the area as a function of a, then find the maximum value of that function.)



The main problems with this one are (a) it takes several steps and (b) the algebra is pretty messy. So let's see.

Step 1 We need to find the equation of the tangent line to e^{-x} at a generic point a. Well, if $f(x) = e^{-x}$ then $f'(x) = -e^{-x}$, so that the line we want has slope $-e^{-a}$. It also goes through the point on the curve, which has coordinates (a, e^{-a}). Using the point-slope form of the equation of a line, we see that the tangent line is

$$y - e^{-a} = -e^{-a}(x - a),$$

which simplifies to

$$y = -e^{-a}x + (a + 1)e^{-a}.$$

Step 2 In order to find the area of the triangle, we need to know at what points this line cuts the two axes. For the y-axis, we need to set $x = 0$, and so we get $y = (a + 1)e^{-a}$. For the x-axis, we set $y = 0$ and solve for x to get $x = a + 1$.

(This is remarkable, by the way: it says that at any point a, if you draw a tangent line to e^{-x} at that point, it will intersect the x-axis one unit forward from where a is. Remember that e^{-x} gets pretty flat as x gets bigger. . .)

Step 3 Now we can find the area of the triangle:

$$A = \frac{1}{2}(a + 1)(a + 1)e^{-a} = \frac{1}{2}(a + 1)^2 e^{-a}.$$

Step 4 The problem asks us to find the a that will make that area as large as possible. In other words it's asking for a maximum. At a maximum, the derivative will be zero. So now we treat a as a variable and find the derivative of the area:

$$A' = \frac{1}{2}\left[2(a + 1)e^{-a} + (a + 1)^2(-e^{-a})\right] = \frac{1}{2}(1 - a^2)e^{-a}.$$

(Do the algebra!) Since e^{-a} is never zero, this derivative is zero only if $a = \pm 1$. So the maximum is probably when $a = 1$.

To check that it really is a maximum, notice that $1-a^2$ is positive for $0 < a < 1$ and negative for $a > 1$. So the area increases as a approaches 1, then decreases. That's clearly a maximum.

Conclusion The largest triangle is the one in the picture, for which $a = 1$. The vertices are at $(0, 0)$, $(0, 2/e)$, and $(2, 0)$, and the area is $2/e$.