

MA121, Spring 2008 — Problem Set 8 Solutions

I. Problems from the textbook:

a. Review problems for chapter 6 (p. 306): 12, 14, 20, 36, 42, 46, 60.

$$12. \int \left(\sqrt{x^3} - \frac{2}{x} \right) dx = \int (x^{3/2} - 2x^{-1}) dx = \frac{2}{5}x^{5/2} - 2\ln(x).$$

$$14. \int 2^x dx = \int e^{\ln(2)x} dx = \frac{1}{\ln(2)} e^{\ln(2)x} = \frac{1}{\ln(2)} 2^x.$$

$$20. \int (3 \cos(t) + 3\sqrt{t}) dt = 3 \sin(t) + 2t^{3/2}$$

36 Hey, a problem where the +C matters! First,

$$\int \sin(x) = -\cos(x),$$

so $F(x) = -\cos(x) + C = C - \cos(x)$. We want

$$4 = F(0) = C - \cos(0) = C - 1,$$

so $C = 5$. Thus, $F(x) = 5 - \cos(x)$.

42 The point is that we want the total area, so we need to compute the areas of both pieces and add. For the first piece, the integral will give the negative of the area, so we integrate and then change sign. So the area is

$$-\int_0^1 (3x^2 - 3) dx + \int_1^3 (3x^2 - 3) dx = 22.$$

Notice that this amounts to computing the integral of the *absolute value* of $3x^2 - 3$.

46 To find the points of intersection, note that both $x(x - \pi)$ and $\sin(x)$ are zero at $x = 0$ and $x = \pi$ (thank goodness, otherwise it'd be very hard to find the intersections). So the region in question is bounded below by $x(x - \pi)$, above by $\sin(x)$, and goes from $x = 0$ to $x = \pi$.

Now there are two ways to think about it. We could do it as we did the previous problem, to get that the area is

$$\int_0^\pi \sin(x) dx - \int_0^\pi (x^2 - \pi x) dx = 2 + \frac{\pi^3}{6}.$$

Alternatively, we could imagine splitting the picture into lots of thin vertical almost-rectangles. The height of each such integral is equal to the *difference* between the two functions. So the area is the sum of the heights times dx , i.e., it is

$$\int_0^{\pi} (\sin(x) dx - (x^2 - \pi x)) dx,$$

which of course gives the same answer.

60 This was tricky. Here's how I thought about it. First of all, graphs A and B are always positive, and graph C is always negative. But graph C is decreasing at first, then increases. So the graph of the derivative of C would have to start negative, then become positive. So it's not in the picture. That means C is the graph of f' .

OK, if C is f' , then the function f is certainly positive. But the integral of a positive function is necessarily *increasing*. (You can see that either by noting that the derivative, which is the original function, is positive, or by thinking of areas under the curve, which grow as x moves to the right.) Either way, the graph of $\int_0^x f(t) dt$ must be an increasing function, hence must be A.

So f is B. As a sanity check, notice that B has an inflection where C has a minimum, which is as it should be. You can also (barely) see that there are maximum and minimum values of f at the two endpoints.

b. Section 7.1, problems 4, 6, 8, 10, 16, 18, 26, 54, 56, 58, 76, 80.

4. $\int e^{-x} dx = -e^{-x}$

6. $\int t \cos(t^2) dt = \left[\begin{array}{l} u = t^2 \\ du = 2t dt \end{array} \right] = \frac{1}{2} \int \cos(u) du = \frac{1}{2} \sin(u) = \frac{1}{2} \sin(t^2)$

8. $\int \sin(3 - t) dt = \cos(3 - t)$

10. Take $u = y^2 + 5$, $du = 2y dy$, so

$$\int y(y^2 + 5)^8 dy = \frac{1}{2} \int u^8 du = \frac{1}{2} \cdot \frac{u^9}{9} = \frac{(y^2 + 5)^9}{18}$$

16. Take $u = 2t - 7$, $du = 2dt$, so

$$\int (2t - 7)^{73} dt = \frac{1}{2} \int u^{73} du = \frac{1}{2} \cdot \frac{u^{74}}{74} = \frac{1}{148} (2t - 7)^{74}$$

18. Take $u = 4 - x$, $du = -dx$ to get

$$\int \frac{1}{\sqrt{4-x}} dx = -2\sqrt{4-x}$$

26. Remember that $\tan(2x) = \frac{\sin(2x)}{\cos(2x)}$. Then take $u = \cos(2x)$, $du = -2 \sin(2x) dx$, so

$$\int \tan(2x) dx = \int \frac{\sin(2x)}{\cos(2x)} dx = -\frac{1}{2} \int \frac{1}{u} du = -\frac{1}{2} \ln(u) = -\frac{1}{2} \ln(\cos(2x))$$

54. The crucial substitution is, of course, $u = \pi x$, $du = \pi dx$. So

$$\int_0^{1/2} \cos(\pi x) dx = \frac{1}{\pi} \int_0^{\pi/2} \cos(u) du = \frac{1}{\pi} (\sin(\pi/2) - \sin(0)) = \frac{1}{\pi}.$$

56. Take $u = x^2$, $du = 2x dx$, so that

$$\int_1^2 2xe^{x^2} dx = \int_1^4 e^u du = e^4 - e.$$

58. Take $u = t + 2$, $du = dt$, to get

$$\int_{-1}^{e-2} \frac{1}{t+2} dt = \int_1^e \frac{1}{u} du = \ln(e) - \ln(1) = 1.$$

76. Both are easy exercises in substitution as long as you remember to change the endpoints. The first integral is 10, the second is 5.

People seem to have gotten confused because they didn't see that the value of a definite integral does not depend on the name of the variable; see comments below on problems 3 and 4.

80. Just integrate; the area is

$$\int_0^2 xe^{x^2} = \left[\begin{array}{l} u = x^2 \\ du = 2x dx \end{array} \right] = \frac{1}{2} \int_0^4 e^u du = \frac{1}{2}(e^4 - 1).$$

c. Section 7.2, problems 2, 4, 8, 12, 14.

The first few should all be straightforward by now, so I'm just writing out the answers. I'm also not going to bother with +C when doing indefinite integrals.

2.

$$\int t \sin t \, dt = -t \cos t + \int \cos t \, dt = -t \cos t + \sin t.$$

4.

$$\int t e^{5t} \, dt = \frac{1}{5} t e^{5t} - \frac{1}{5} \int e^{5t} \, dt = \frac{1}{5} t e^{5t} - \frac{1}{25} e^{5t}.$$

(Or make $u = 5t$ first, then use integration by parts.)

8.

$$\int y \ln(y) \, dy = \frac{y^2}{2} \ln(y) - \int \frac{y^2}{2} \frac{1}{y} \, dy, \quad du = \frac{1}{2} y^2 \ln(y) - \frac{1}{2} \frac{y^2}{2} = \frac{2y^2 \ln(y) - y^2}{4}.$$

12. This one needs a dastardly trick, namely, one has to remember that $\cos^2 \theta = 1 - \sin^2 \theta$. Doing straight integration by parts gives

$$\int \sin^2 \theta \, d\theta = \int \sin \theta \sin \theta \, d\theta = -\cos \theta \sin \theta + \int \cos \theta \cos \theta \, d\theta.$$

In other words:

$$\begin{aligned} \int \sin^2 \theta \, d\theta &= -\sin \theta \cos \theta + \int \cos^2 \theta \, d\theta \\ &= -\sin \theta \cos \theta + \int (1 - \sin^2 \theta) \, d\theta \\ &= -\sin \theta \cos \theta + \theta - \int \sin^2 \theta \, d\theta. \end{aligned}$$

Moving the last integral to the left-hand-side of the equation gives

$$2 \int \sin^2 \theta \, d\theta = -\sin \theta \cos \theta + \theta,$$

so that, finally,

$$\int \sin^2 \theta \, d\theta = \frac{\theta}{2} - \frac{1}{2} \sin \theta \cos \theta = \frac{\theta}{2} - \frac{1}{4} \sin 2\theta.$$

(For the last step, remember that $\sin(2\theta) = 2 \sin \theta \cos \theta$. It may or may not be useful to make that change, but it makes the answer shorter.)

14. You can do this in one step if you know the anti-derivative of $\ln(t)$. I didn't, so I did it with two integrations by parts:

$$\begin{aligned} \int (\ln t)^2 dt &= \int 1 \cdot (\ln t)^2 dt = t(\ln t)^2 - \int t \cdot 2 \ln(t) \frac{1}{t} dt \\ &= t(\ln t)^2 - \int 2 \ln t dt \\ &= t(\ln t)^2 - 2t \ln t + \int 2 dt \\ &= t(\ln t)^2 - 2t \ln t + 2t \end{aligned}$$

Alternatively, one could let $u = \ln(t)$. Then $t = e^u$, so $dt = e^u du$. Doing the substitution gives $\int u^2 e^u du$, which again can be done with two integrations by parts.

2. To compute the integral $\int_2^4 \frac{1 - (\frac{x}{2})^2}{x} dx$, we try the substitution $u = x/2$. Which of the following do we get?

- A. $\int_1^2 \frac{1 - u^2}{u} du$ B. $\int_1^2 \frac{1 - u^2}{4u} du$ C. $\int_2^4 \frac{1 - u^2}{u} du$
D. $\int_2^4 \frac{1 - u^2}{2u} du$ E. $\int_1^2 \frac{1 - u^2}{2u} du$

We let $u = x/2$, so $du = \frac{1}{2}dx$, which we can rewrite as $x = 2u$, $dx = 2du$ if we like. The endpoints are obtained by plugging $x = 2$ and $x = 4$ in the formula for u . So we get

$$\int_2^4 \frac{1 - (\frac{x}{2})^2}{x} dx = \int_1^2 \frac{1 - u^2}{2u} 2 du = \int_1^2 \frac{1 - u^2}{u} du.$$

So A is the right choice.

This was a test problem last year.

3. Show that

$$\int_a^b f(x) dx = \int_{a+c}^{b+c} f(x - c) dx$$

by choosing the right substitution. Draw a picture to explain what's going on.

This should have been easy. . .

$$\int_{a+c}^{b+c} f(x-c) dx = \left[\begin{array}{l} u = x - c \\ du = dx \end{array} \right] = \int_a^b f(u) du,$$

and we're done.

Some people got confused because the variable in this integral is u , not x . But it's the same function, integrated over the same range of values! Or, if you trust the formalism more than "looking," just do it:

$$\int_a^b f(u) du = \left[\begin{array}{l} x = u \\ dx = du \end{array} \right] = \int_a^b f(x) dx.$$

The picture would show two identical graphs located at different points: the graph of $y = f(x - c)$ is just the graph of $y = f(x)$ displaced c units to the right.

4. Show that

$$\frac{1}{c} \int_{ca}^{cb} f(x) dx = \int_a^b f(cx) dx$$

by choosing the right substitution.

(Notice that I have corrected the typo!)

This is very similar to the other, except that scaling the variable stretches the graph, so we get a scaling factor for the integrals too. It's easier to start from the second one:

$$\int_a^b f(cx) dx = \left[\begin{array}{l} u = cx \\ du = c dx \end{array} \right] = \frac{1}{c} \int_{ca}^{cb} f(u) du,$$

and we're done.

Notice that when you want to prove that two things are equal the best approach is always to start with one of them and follow a chain of equal signs until you get to the other. It's not always that easy, but when you can do it that way that's what you should do.

5. Check that

$$\frac{1}{1-t} + \frac{1}{1+t} = \frac{2}{1-t^2}.$$

Use this to find an antiderivative for

$$f(t) = \frac{1}{1-t^2}.$$

(This gives you a glimpse of a method for computing integrals that is known as "partial fraction decomposition.")

This is also very easy; the point was to show you an example of the method.

$$\int \frac{1}{1-t^2} dt = \frac{1}{2} \left(\int \frac{1}{1-t} dt + \int \frac{1}{1+t} dt \right) = \frac{1}{2} (-\ln(1-t) + \ln(1+t)).$$

Show-offs might have written this as

$$\frac{1}{2} \ln \left(\frac{1+t}{1-t} \right),$$

or even as

$$\ln \left(\sqrt{\frac{1+t}{1-t}} \right).$$

Real show-offs would have added that this only works for $-1 < t < 1$.