First Integrals With Answers

I hope these answers are right. In any case, you can check them easily with your calculator and/or *Mathematica*.

1. Easier ones. These mostly can be done by guessing and checking, though in a couple of cases it's easier to do a substitution. Notice that these are all *definite* integrals, so the answer is a number.

a. $\int_0^3 x\,dx = \left[\frac{x^2}{2}\right]_0^3 = \frac{9}{2}$

b. $\int_0^3 (5x+1)\,dx = 5\int_0^3 x\,dx + \int_0^3 dx = 5\cdot\frac{9}{2} + 3 = \frac{51}{2}$

c. $\int_0^{2\pi} \cos x\,dx = \left[\sin(x)\right]_0^{2\pi} = 0$

d. $\int_0^{\pi/2} \sin x\,dx = \left[-\cos(x)\right]_0^{\pi/2} = 0 - (-1) = 1$

e. $\int_{-1}^1 x^2\,dx = \left[\frac{x^3}{3}\right]_{-1}^1 = \frac{1}{3} - \frac{-1}{3} = \frac{2}{3}$

f. $\int_{-1}^1 x^3\,dx = 0$ (odd function!)

g. $\int_1^3 \frac{1}{x}\,dx = \left[\ln(x)\right]_1^3 = \ln(3)$

h. $\int_0^a e^{-x}\,dx = [e^{-x}]_0^a = 1 - e^{-a}$

i. $\int_1^4 \frac{1}{\sqrt{x}}\,dx = \int_1^4 x^{-1/2}\,dx = \left[\frac{x^{1/2}}{1/2}\right]_1^4 = 2\left[\sqrt{x}\right]_1^4 = 2(\sqrt{4}-1) = 2$

j. $\int_1^4 \sqrt{x}\,dx = \int_1^4 x^{1/2}\,dx = \left[\frac{x^{3/2}}{3/2}\right]_1^4 = \frac{2}{3}(4^{3/2}-1) = \frac{2}{3}(8-1) = \frac{14}{3}$

k. If $p \neq -1$, $\int_a^b x^p\,dx = \left[\frac{x^{p+1}}{p+1}\right]_a^b = \frac{b^{p+1}-a^{p+1}}{p+1}$

l. $\int_0^2 \sin(3x)\,dx = \left[-\frac{1}{3}\cos 3x\right]_0^2 = -\frac{1}{3}(\cos(6)-1) = \frac{1-\cos(6)}{3} = 0.0132766\ldots$

m. $\displaystyle\int_0^2 \frac{x}{\sqrt{x^2+1}}\,dx = \begin{bmatrix} u = x^2+1 \\ du = 2x dx \end{bmatrix} = \int_1^5 \frac{1}{2\sqrt{u}}\,du = \left[\sqrt{u}\right]_1^5 = \sqrt{5}-1$

n. $\displaystyle\int_1^3 \frac{1}{x+1} = \Bigg[\ln(x+1)\Bigg]_1^3 = \ln(4)-\ln(2) = \ln(\frac{4}{2}) = \ln(2)$

o. $\displaystyle\int_1^3 \frac{x-1}{x+1} = \int_1^3 \left(\frac{x+1}{x+1} - \frac{2}{x+1}\right)\,dx = 2 - 2\ln(2)$

2. Harder ones. Most of these require either an algebraic trick or a substitution, or both. The last one requires a substitution and then an integration by parts!

In some cases, after a substitution we get an integral we've already done. In those cases, I've used the previous result without comment.

a. $\displaystyle\int \frac{1}{\sqrt{x-1}+\sqrt{x+1}}\,dx$. Multiply numerator and denominator by the conjugate radical, i.e., $\sqrt{x-1}-\sqrt{x+1}$:

$$\int \frac{1}{\sqrt{x-1}+\sqrt{x+1}} = \int \frac{\sqrt{x-1}-\sqrt{x+1}}{(x-1)-(x+1)}\,dx = \frac{1}{2}\int \left(\sqrt{x+1}-\sqrt{x-1}\right)\,dx$$

$$= \frac{1}{3}\left((x+1)^{3/2}-(x-1)^{3/2}\right)$$

b. $\displaystyle\int e^x \sin(e^x)\,dx = \begin{bmatrix} u = e^x \\ du = e^x dx \end{bmatrix} = \int \sin(u)\,du = -\cos(u) = -\cos(e^x)$

c. $\displaystyle\int xe^{-x^2}\,dx = \begin{bmatrix} u = -x^2 \\ du = -2x dx \end{bmatrix} = -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u = -\frac{1}{2}e^{-x^2}$

d. $\displaystyle\int \frac{\ln(x)}{x}\,dx = \begin{bmatrix} u = \ln(x) \\ du = \frac{dx}{x} \end{bmatrix} = \int u\,du = \frac{1}{2}u^2 = \frac{1}{2}(\ln(x))^2$

e. $\displaystyle\int \ln(\cos(x))\tan(x)\,dx = \int \ln(\cos(x))\frac{\sin x}{\cos x} = \begin{bmatrix} u = \cos(x) \\ du = \sin(x) dx \end{bmatrix}$

$$= \int \frac{\ln(u)}{u}\,du = \frac{1}{2}(\ln(u))^2 = \frac{1}{2}(\ln(\cos(x)))^2$$

f. $\displaystyle\int \frac{\ln(\ln(x))}{x\ln(x)}\,dx = \begin{bmatrix} u = \ln(x) \\ du = \frac{dx}{x} \end{bmatrix} = \int \frac{\ln(u)}{u}\,du = \frac{1}{2}(\ln(u))^2 = \frac{1}{2}\left(\ln(\ln(x))\right)^2$

g. $\displaystyle\int \frac{1}{1+\sqrt{x+1}}\,dx.$

This one is pretty nasty; we'll take $u = 1+\sqrt{x+1}$. Then $du = \dfrac{dx}{2\sqrt{x+1}}$; solving for dx we get $dx = 2\sqrt{x+1}du = 2(u-1)du$. So...

$$\int \frac{1}{1+\sqrt{x+1}}\,dx = \begin{bmatrix} u = 1+\sqrt{x+1} \\ dx = 2(u-1)du \end{bmatrix} = \int \frac{2u-2}{u}\,du$$

$$= \int 2\,du - \int \frac{2}{u}\,du = 2u - 2\ln(u) = 2(1+\sqrt{x+1}) - 2\ln(1+\sqrt{x+1})$$

Notice that since constants don't matter, we could have also given the answer as $2\sqrt{x+1} - 2\ln(1+\sqrt{x+1})$, which is what *Mathematica* gives.

h. $$\int \frac{1}{1+e^x}\,dx = \int \frac{e^{-x}}{e^{-x}+1}\,dx = \begin{bmatrix} u = e^{-x} \\ du = -e^{-x}dx \end{bmatrix} = -\int \frac{1}{u+1}\,du$$

$$= -\ln(1+u) = -\ln(1+e^{-x})$$

i. $$\int e^{\sqrt{x}}\,dx = \begin{bmatrix} u = \sqrt{x} \\ du = \frac{1}{2\sqrt{x}dx} \end{bmatrix} = \begin{bmatrix} x = u^2 \\ dx = 2u du \end{bmatrix} = \int 2ue^u\,du$$

$$= 2\int ue^u\,du = 2ue^u - 2\int e^u\,du = 2ue^u - 2e^u$$