MA121, Spring 2008 — Midterm 1 Solutions

The quotation from Barnes Wallis on the first page is worth commenting on. Wallis is quite famous in England as an inventor, among other things, of the "bouncing bomb," which proved important in World War II. The book from which I took the quote is called *Mathematics with Love*. It collects the letters that Wallis wrote to Molly Bloxam, later to be his wife. The story is that Bloxam's father forbade the two to court, but agreed to let Wallis write her letters about calculus, which she was trying to learn. So Wallis did, keeping in contact via mathematics. The restrictions on their interaction were slowly relaxed, and one can follow the evolution of their courtship in the book. It's lots of fun, and I recommend it.

1. [20 **points**] Suppose p is the price of gasoline and $f(p)$ is the number of gallons of gasoline sold over a weekend when the price is p. It is natural to guess that if the price increases people will buy less gasoline. On the other hand, it seems likely that folks will not be able to reduce their consumption below a certain minimum. Translate these claims into mathematical statements about the function f, its derivative, etc. Explain your reasoning.

The hardest thing about this problem, from what I observed on Tuesday, was that folks couldn't believe it wanted as little as it did.

Anyway, what I expected you to say is that

a. $f(p)$ is decreasing, so that we must have $f'(p) < 0$.

b. $f(p)$ is bounded below, and probably tends to some limit as p grows.

c. Therefore, the graph of f must be concave up, and $f''(p)$ must (eventually) be positive.

Notice that $f''(p)$ could be negative at first. Notice also that it is *obvious* that consumption has a lower limit, because it certainly can't be negative. I pointed it out in order to help; I hope it did help.

2. [20 **points**] If we set $y = e^{rx}$, then one can choose the number r such that the differential equation

$$y'' + 5y' - 6y = 0$$

becomes true. Find which value(s) of r will work.

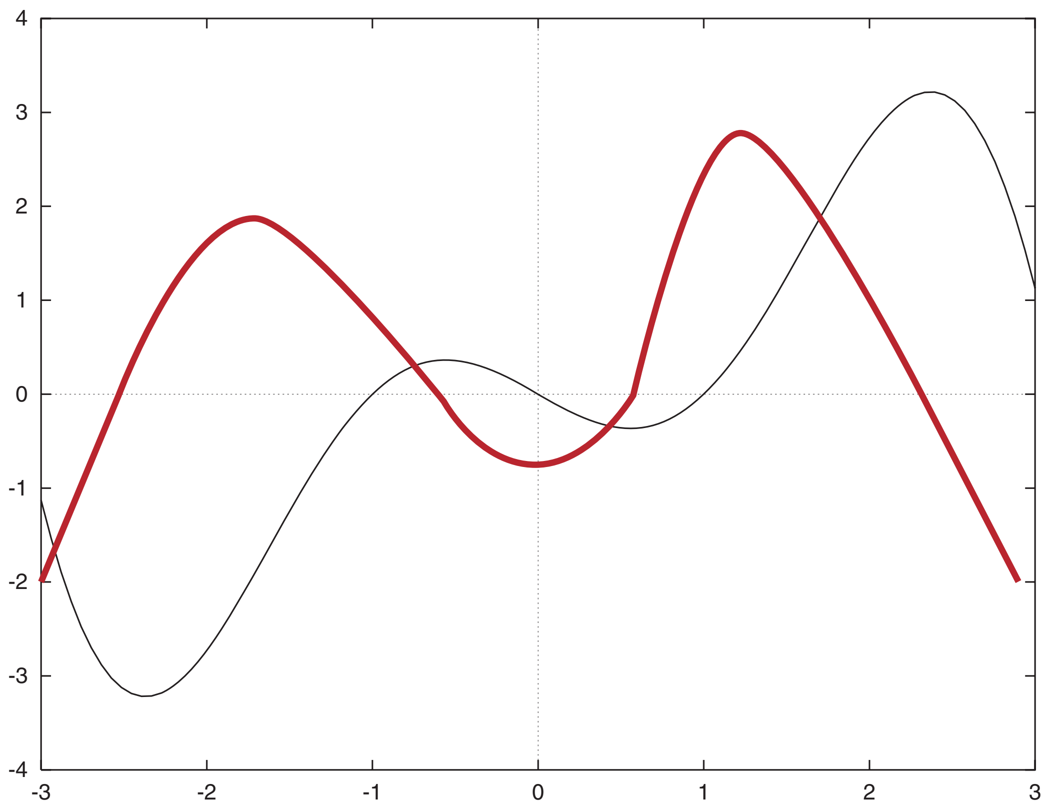
(Start by computing y' and y''.)

Well, I thought finding the derivatives would be the easy part. It doesn't seem to have turned out that way.

Anyway, $y' = re^{rx}$ and $y'' = r^2e^{rx}$, so the equation becomes

$$0 = r^2e^{rx} + 5re^{rx} - 6e^{rx} = (r^2 + 5r - 6)e^{rx}.$$

Since e^{rx} cannot be zero, this will be true ir only if $r^2+5r-6 = 0$, which happens if and only if $r = 1$ or $r = -6$.



3. [20 points] The figure above shows the graph of a function $g(x)$. On the same set of axes, sketch the graph of the derivative $g'(x)$.

It's hard to draw the solution electronically. The attempt above is fairly rough, but should give you the basic idea. The red curve is $g'(x)$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-2	1	3	2	0
-1	0	2	-1	-1
0	1	2	-2	3
1	2	1	0	2
2	3	0	0	-1

4. [20 **points**] Suppose f and g are functions and $h(x) = 3x^2 + f(g(x))$. Using the table of values given above, find $h'(1)$.

Well, the first part is easy, but the second requires that you know the chain rule. So $h'(x) = 6x + f'(g(x))g'(x)$. Plugging in $x = 1$ gives

$$h'(1) = 6 + f'(g(1))g'(1).$$

From the table, $g(1) = 0$, so this becomes

$$h'(1) = 6 + f'(0)g'(1) = 6 + 2 \times 2 = 10.$$

5. [20 **points**] Let $f(x) = \sqrt{x^3 + 1}$. Notice that $f(2) = \sqrt{9} = 3$.

a. Find the equation of the line that is tangent to the graph of $y = f(x)$ at the point $(2, 3)$.

The derivative is

$$f'(x) = \frac{3x^2}{2\sqrt{x^3 + 1}}.$$

So the slope of the tangent line we want is $f'(2) = 2$. The line goes through the point $(2, 3)$, so the equation we want is

$$y - 3 = 2(x - 2)$$

or

$$y = 2x - 1.$$

b. If you try to approximate $f(2.1)$ using the tangent line you just found, will the result be an underestimate or an overestimate? Why?

The most straightforward approach was just to compute $f''(2)$ and see that it is positive. Hence, the graph of f is concave up, so it lies *above* the tangent line. So the approximation given by the tangent line is an underestimate.

You could also have found the concavity using your calculator to draw a graph, though it is so steep that seeing it correctly requires a little bit of fiddling. You should know how to use your tools well!

Bonus Problem [10 points] Suppose you have a function f which is continuous for $a \leqslant x \leqslant b$ and differentiable for $a < x < b$.

a. What does the Mean Value Theorem say about the function f?

It says that

$$f(b) - f(a) = f'(\xi)(b - a),$$

where ξ is some number between a and b.

b. Why does Fernando like this theorem so much?

Not because it is hard to prove (it is not!), but because it contains the crucial fact about derivatives: that knowing the derivative tells you something about how the function grows. I also enjoy the fact that no one realized a proof was necessary until the 19th century, when Cauchy found himself writing up his lecture notes.