MA357, Spring 2008 — Problem Set 2

This problem set focuses on divisibility and the greatest common divisor. The problems are intended to require some thought. A write-up of all the problems except the ones labelled "to explore" (about which see below) is due on **Friday, February 22**.

Beginning with this problem set, I will always include at least one problem labelled "to explore." These problems are suggestions for open-ended explorations that may or may not be easy to do but which I think are interesting. Eventually, you will be asked to choose one of the "to explore" problems and write it up; for now, they are offered for your enjoyment only, and you need not write them up or turn them in.

1. Suppose you know that $\gcd(a, b) = 1$. What can you say about each of the following?

a. $\gcd(a+b, a-b)$

b. $\gcd(a, a+b)$

c. $\gcd(2a, 2b)$

d. $\gcd(2a, b)$

2. Let $n \geq 1$ be an integer. Prove that $n! + 1$ and $(n+1)! + 1$ are always relatively prime.

3. Prove that

a. the sum of two odd numbers is even,

b. the square of an odd number is always of the form $4k + 1$, and

c. if n is *not* divisible by 3, then $n^2 - 1$ *is* divisible by 3.

(Hint: each of these depends on using division with remainder.)

4. Use a divisibility argument to show that the equation $x^2 = 2y^2$ has no solutions in integers other than $x = y = 0$. (This implies that $\sqrt{2}$ is irrational.)

Hint: The easiest way to assume that two non-zero integers x and y exist such that $x^2 = 2y^2$, and then show that this leads to a contradiction.

5. Find the gcd of 771769 and 32378, and express it as a linear combination of these two numbers.

6. One egg timer can time an interval of exactly 5 minutes, and a second can time an interval of exactly 11 minutes. How can we boil an egg for exactly 3 minutes (without buying another timer)?

7. Suppose $x = \frac{u}{v}$ (in lowest terms) is a solution of the equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0.$$

Show that $u | a_0$ and $v | a_n$.

8. In 1509, DeBouvelles claimed that for every $n \geq 1$ at least one of the numbers $6n - 1$ and $6n + 1$ was prime. Find a counterexample to show that he was wrong, then show that there are infinitely many counterexamples (i.e., show that there are infinitely many n such that both $6n - 1$ and $6n + 1$ are composite).

9. Find all the integer solutions of $15x + 7y = 310$, and then decide how many of them are *positive* integer solutions.

10. Let

$$S = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}.$$

Prove that S is not an integer. (Hint: show that the denominator is divisible by 2.)

To Explore: Suppose m and n are positive integers and we use the Euclidean algorithm to compute their gcd. If m and n are big, we should try to estimate how many steps the algorithm is going to take before we launch into it. Try to find a way to estimate the maximum number of steps that will be necessary. (See problem 5.3 in Silverman, but note that the result he suggests is *not* the best possible one.)

Hint: You have probably heard of the *Fibonacci Sequence*, defined by $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-1}$ for every $n \geq 2$. One way to attack the problem of estimating the number of steps needed in the Euclidean algorithm is to show that the case of two successive terms in this sequence is a kind of "worst possible case."

To Explore: Silverman, problems 5.5 and 5.6.

To Explore: The locker problem from Assignment 1 has the potential to expand into many variants. For example, you might specify the set of lockers you want to be open at the end, and ask for a sequence of students that will achieve that. For example, for what sequence of students will it be true that in the end only locker number 1 is open? (Assume that you're allowed to send student number d through the row of lockers more than once.)

Another variant is to replace the lockers with other objects that have more than two states. For example, what if we used lamps with pull-cords that could be in four states: off, dim, medium, bright. The lamps would cycle through these states as the students pulled the cords. Assuming that student d pulls cord n when d divides n, what can you say about the set of lamps that end up in each state? Can you work out sets of students that will achieve results specified in advance?