RIGID CALABI-YAU THREEFOLDS OVER $\mathbb{Q}$ ARE MODULAR: A FOOTNOTE TO SERRE

FERNANDO Q. GOUVÊA AND NORIKO YUI

ABSTRACT. The proof of Serre's conjecture on Galois representations over finite fields allows us to show, using a method due to Serre himself, that all rigid Calabi-Yau threefolds defined over $\mathbb{Q}$ are modular.

The safest general characterization of the
European philosophical tradition is that it
consists of a series of footnotes to Plato.
Alfred North Whitehead,
Process and Reality

In the mid-1980s, J.-P. Serre conjectured in [9] that all absolutely irreducible odd two-dimensional representations of $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ over a finite field come from modular forms of prescribed weight, level, and character. This has now been proved by C. Khare and J.-P. Wintenberger; see [6, 7]. Because this result can be seen as a generalization of Artin Reciprocity to the GL_2 case (over $\mathbb{Q}$), we will refer to it as "Serre Reciprocity."

Already in [9], Serre showed how, given a compatible system of ℓ-adic Galois representations and bounds on the weight and level of the predicted modular forms in characteristic ℓ, one can use Serre Reciprocity to obtain results in characteristic zero. We refer to this as "Serre's method" and describe it in Section 1 below.

The goal of this paper is to use Serre's method to show that certain geometric Galois representations are modular. Specifically, we show that the representation obtained from the third étale cohomology of a rigid Calabi-Yau threefold defined over $\mathbb{Q}$ comes from a modular form of weight 4 on $\Gamma_0(N)$. The proof is an immediate application of Serre's method; it can, in fact, be read off directly from [9, Section 4.8], which is why describe this short paper as a "footnote to Serre."

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The question of the modularity of the Galois representations obtained from Calabi-Yau threefolds over $\mathbb{Q}$ has been much studied, and a large number of examples are now available; see [8] for a survey. Since current technology restricts us to low-dimensional Galois representations, many of the examples involve rigid Calabi-Yau threefolds, defined below, simply because in that case the representation is automatically of dimension two. In addition to the many examples, Dieulefait has shown in [2] that if X has good reduction at small primes then it is modular. Richard Taylor showed in [10] that rigid Calabi-Yau manifolds over $\mathbb{Q}$ are *potentially modular*, i.e., that there exists a totally real field F such that the restrictions to $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ of the representations ρ_ℓ are attached to automorphic representations over F.

As we will indicate below, the same methods also apply to the non-rigid case if one can isolate an irreducible two-dimensional "piece" of the cohomology. In this case, we obtain modular forms of weight 4 and of weight 2, in agreement with many examples found by Meyer and others. In general, however, the middle cohomology groups of non-rigid Calabi-Yau threefolds do not decompose into products of two-dimensional pieces.

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1. Serre Reciprocity

Let $G = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be the absolute Galois group of $\mathbb{Q}$, let $\mathbb{F}$ be a finite field of characteristic ℓ, and let $\rho : G \to \mathrm{GL}_2(\mathbb{F})$ be an absolutely irreducible representation. We will assume throughout that ρ is *odd*, that is, we will assume that if $c \in G$ is (any) complex conjugation, we have $\det \rho(c) = -1$. We will let S be the finite set of primes such that ρ is unramified at all primes not in S.

In [9], Serre associated to any such ρ a triple (N, k, ε), where N and k are positive integers, $k \geq 2$, and ε is a Dirichlet character modulo N. We will briefly recall below how this triple is obtained, but we refer the reader to [9] and [3] for details.

The result conjectured in [9] and proved in [6, 7] is:

Theorem 1 (Serre–Khare–Wintenberger)**.** *Let G, ℓ, and $\mathbb{F}$ be as above. Suppose $\rho : G \to \mathrm{GL}_2(\mathbb{F})$ is an odd absolutely irreducible representation, and let (N, k, ε) be the Serre parameters attached to ρ. Then there exist:*

- *a cuspidal modular eigenform f on $\Gamma_0(N)$, of weight k and character ε and defined over a number field K, and*
- *a prime λ of K with residue field $\mathbb{F}$*

such that the reduction modulo λ of the λ-adic representation attached to f is isomorphic to ρ.

Part of the power of this result comes from the fact that the triple (N, k, ε) is specified in advance in terms of ρ, which restricts us to a finite number of possibilities for the eigenform f. It will be helpful to recall how these parameters are obtained.

The level N is fairly easy to describe: it is the prime-to-ℓ part of the Artin conductor of the representation ρ. As such, it is divisible only by primes $p \in S$, $p \neq \ell$. If we set

$$N = \prod_{p \in S} p^{e(p)},$$

the exponent $e(p)$ is entirely determined by the image of the decomposition group at p. In particular, it is useful to note that if ρ is *tamely* ramified at p then $e(p) = 1$.

The weight k is the most delicate of the three parameters. It depends only on the image of the decomposition group at ℓ, but the recipe for computing k is complicated; see [9] and [3] for details.

Note that as proved by Khare and Winterberger the result actually obtains a modular form defined over the finite field $\mathbb{F}$. Such forms will lift to characteristic zero provided $k \neq 1$. Because we need to be sure that we are dealing with modular forms that do lift to characteristic zero, we will use Serre's normalization of the weight. In particular, the argument below will not work when the "natural" choice of weight is $k = 1$. See [5], however, for how this case can be handled.

Finally, the character ε is determined by the formula

$$\det \rho = \varepsilon \chi_\ell^{k-1},$$

where χ_ℓ is the (reduction mod ℓ of the) ℓ-adic cyclotomic character. Notice that this formula determines k modulo $\ell - 1$.

While Serre Reciprocity is a statement about Galois representations over finite fields, it can often be used to show the modularity of representations in characteristic zero as well. The idea is due to Serre himself; it will perhaps be useful to have a formalized version of it.

Theorem 2 (Serre's Method)**.** *Fix a number field K, and let λ run over primes of K. For each λ, let K_λ denote the completion of K at λ and let $\kappa(\lambda)$ be the residue field. Let ℓ be the characteristic of $\kappa(\lambda)$.*

Fix a finite set S of primes in $\mathbb{Q}$. For each $p \notin S$, let Frob_p *be a choice of arithmetic Frobenius element in* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Suppose we have, for each λ, a two-dimensional K_λ vector space V_λ with an action of $G = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. This gives a representation ρ_λ. Assume ρ_λ is odd and unramified outside $S \cup \{\ell\}$.

For each λ we can find a G-stable lattice, reduce modulo λ, and semisimplify if necessary to obtain a semisimple odd two-dimensional Galois representation

$$\overline{\rho}_\lambda : G \longrightarrow \mathrm{GL}_2(\kappa(\lambda))$$

unramified outside $S \cup \{\ell\}$.

Fix an infinite set I of primes in K. Suppose we can show that:

(1) *For all $\lambda \in I$, the representation $\overline{\rho}_\lambda$ is absolutely irreducible.*
(2) *There exists a family $Q_p(X) = X^2 - A_p X + D_p \in K[X]$ of polynomials of degree 2, indexed by primes $p \in \mathbb{Q}$, $p \notin S$, such that for all $\lambda \in I$ and (given λ) all $p \notin S \cup \{\ell\}$, the characteristic polynomial of* Frob_p *acting on V_λ is equal to $Q_p(X)$.*
(3) *There exists an integer k_0 such that for all $\lambda \in I$ the Serre weight k attached to $\overline{\rho}_\lambda$ is satisfies $1 < k \leq k_0$.*
(4) *There exists an integer N_0 such that for all $\lambda \in I$ the Serre level N attached to $\overline{\rho}_\lambda$ is a divisor of N_0.*

Then there exists a cuspidal Hecke eigenform form f (new of level dividing N_0, weight less than or equal to k_0, defined over K) such that for all λ the λ-adic representation $\rho_{f,\lambda}$ attached to f is isomorphic to ρ_λ.

Proof. Choose $\lambda \in I$ and apply Serre Reciprocity to $\overline{\rho}_\lambda$. We get an eigenform f_λ of weight k less than or equal to k_0, character ε, and level dividing N_0. A priori, f may be defined over an extension of K whose residue field at a prime λ' over λ is still $\kappa(\lambda)$. The fact that f_λ corresponds to ρ_λ tells us that

$$A_p \equiv a_p(f_\lambda)(mod\ \lambda')$$

and

$$D_p \equiv \varepsilon(p)p^{k-1}(mod\ \lambda')$$

for all $p \notin S \cup \{\ell\}$.

Since the set of all eigenforms of weight bounded by k_0 and level dividing N_0 is *finite* and there are infinitely many $\lambda \in I$, there must exist a modular form f such that $f_\lambda = f$ for infinitely many λ. But then, for each $p \notin S$ we will have

$$A_p \equiv a_p(f)(mod\ \lambda)$$

and

$$D_p \equiv \varepsilon(p)p^{k-1}(mod\ \lambda)$$

for infinitely many λ, which implies $A_p = a_p$ for all $p \notin S$. This is enough to show that f is the eigenform we wanted to find and implies, in particular, that it has coefficients in K. □

In the case of representations coming from geometry, the representations will typically be obtained from the (dual of the) étale cohomology of an algebraic variety X defined over $\mathbb{Q}$. The field K is then just $\mathbb{Q}$. The set S is then the set of primes of bad reduction for X and the existence of the $Q_p(X)$ follows from the Weil Conjectures as proved by Deligne.

For an example with K a totally real field, consider the case of abelian varieties with real multiplication; see [9, Section 4.7]

2. Modularity of rigid Calabi-Yau threefolds over $\mathbb{Q}$

We want to apply Serre's method to the representation obtained from the middle étale cohomology of a rigid Calabi-Yau threefold defined over $\mathbb{Q}$. We recall the definitions.

Definition 1. *Let X be a smooth projective threefold defined over $\mathbb{C}$. We call a* Calabi-Yau threefold *if*

(1) $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$, *and*
(2) $K_X := \wedge^3\Omega^1_X \simeq \mathcal{O}_X$, *that is, the canonical bundle is trivial.*

As usual, we define the Hodge numbers

$$h^{i,j}(X) := \dim_{\mathbb{C}} H^j(X, \Omega^i_X).$$

By complex conjugation, $h^{i,j}(X) = h^{j,i}(X)$, and by Serre duality, $h^{i,j}(X) = h^{3-j,3-i}(X)$ for $0 \leq i, j \leq 3$. The Hodge decomposition gives

$$h^k(X) = \dim_{\mathbb{C}} H^k(X, \mathbb{C}) = \sum_{i+j=k} h^{i,j}(X).$$

The number $h^k(X)$ is called the k-th Betti number of X and often denoted $B_k(X)$.

If X is Calabi-Yau, then the first condition implies that

$$h^{1,0}(X) = h^{2,0}(X) = 0,$$

and the second condition, together with Serre duality, yields

$$h^{3,0} = h^{0,3} = 1.$$

We can summarize all this by drawing the "Hodge diamond" of X:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1}(X) & & 0 & \\ 1 & & h^{2,1}(X) & & h^{1,2}(X) & & 1 \\ & 0 & & h^{2,2}(X) & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}
\qquad
\begin{array}{l} h^0(X) = 1 \\ h^1(X) = 0 \\ h^2(X) = h^{1,1}(X) \\ h^3(X) = 2(1 + h^{2,1}(X)) \\ h^4(X) = h^{2,2}(X) = h^{1,1}(X) \\ h^5(X) = 0 \\ h^6(X) = 1 \end{array}$$

Calabi-Yau threefolds are Kähler manifolds, so $h^{1,1}(X) > 0$. All 2-cycles on Calabi-Yau threefolds are algebraic, as follows follows from the Lefschetz $(1,1)$ theorem that $H^2(X, \mathbb{Z}) \simeq \mathrm{Pic}(\overline{X})$. In particular, $h^{1,1}(X) = h^2(X) = \mathrm{rk}\,\mathrm{Pic}(\overline{X})$.

Definition 2. *Let X be a Calabi-Yau threefold defined over $\mathbb{C}$. We say that X is* rigid *if $h^{2,1}(X) = h^{1,2}(X) = 0$, so that $h^3(X) = 2$.*

The name "rigid" comes from the fact that the space of deformations of a Calabi-Yau manifold has dimension $h^{1,2}(X)$. There are more than 50 known examples of rigid Calabi-Yau threefolds (up to birational equivalence over $\mathbb{C}$). It is still an open problem to decide whether the number of such examples is finite up to birational transformation over $\mathbb{C}$.

Since we are interested in Galois representations, we focus on Calabi-Yau threefolds defined over $\mathbb{Q}$. Of course, a given Calabi-Yau threefold over $\mathbb{C}$ may well have many different realizations over $\mathbb{Q}$. Notice that the fact that X is a rigid Calabi-Yau manifold is independent of the choice of model over $\mathbb{Q}$, but the Galois representation (and therefore the modular forms we will find) depend strongly on that choice. We will comment further on this below.

Let X be a rigid Calabi-Yau threefold defined over $\mathbb{Q}$. Then X always has a model defined over $\mathbb{Z}$; we assume one has been chosen and fixed. We will apply Theorem 2 with $K = \mathbb{Q}$ and S the (finite) set of primes at which X has bad reduction. Let $\overline{X} = X \otimes \overline{\mathbb{Q}}$. For each prime ℓ in $\mathbb{Q}$, let

$$V_\ell = H^3(\overline{X}, \mathbb{Q}_\ell)^\vee.$$

(We need to dualize because we want to work with the arithmetic Frobenius.) We know that if $p \notin S \cup \ell$, this representation will be unramified at p.

The assumption that X is rigid means that that V_ℓ is two dimensional and that its Hodge decomposition is of the form $(3,0) + (0,3)$. By Pontryagin duality, we have

$$\det \rho_\ell = \chi_\ell^3$$

so that ρ_ℓ is odd. Let $\overline{\rho}_\ell$ be the representation obtained by reducing modulo ℓ.

In [9, Section 4.8], Serre checked that conditions (1) and (2) above hold for sufficiently large ℓ. A theorem of Fontaine (see also [3]) shows that for all large enough ℓ the Serre weight parameter will be $k = 4$. Finally, using a bound on the Artin conductor found in [9, Section 4.9], Serre showed that under certain congruence conditions on ℓ we show that N is a divisor of

$$N_0 = \prod_{p \in S} p^{e(p)},$$

where $e(2) = 8$, $e(3) = 5$, and $e(p) = 2$ for all other primes $p \in S$. We therefore let I be the (infinite) set of primes ℓ that satisfy Serre's congruence conditions. Theorem 2 then gives our result:

Theorem 3. *Let X be a rigid Calabi-Yau threefold defined over $\mathbb{Q}$, and use the notations above. Then there exists a Hecke eigenform f of weight 4, level dividing N_0, and trivial character such that ρ_ℓ is equivalent to $\rho_{f,\ell}$ for all ℓ.*

Notice that while the initial bound on the level is obtained only for ℓ satisfying certain congruence conditions, the final result shows that in fact the bound holds without those conditions, since the level of the final form f bounds the conductor of all the residual representations. Thus we recover the bound obtained by Dieulefait in [1].

In other words, all rigid Calabi-Yau threefolds defined over $\mathbb{Q}$ are modular. In particular, this implies that the L-function corresponding to the third étale cohomology of such a threefold is the same as that of a modular form of weight 4, and hence is holomorphic and satisfies a functional equation relating values at s to values at $4 - s$.

Serre's method is applicable, as he shows in [9], to all odd-dimensional algebraic varieties whose middle-dimensional cohomology is of dimension two.

3. The Non-Rigid Case

The reason to focus on the rigid case is, of course, that we get a Galois representation of dimension two, which should then come from a modular form. Higher dimensional representations should be automorphic, but the type of corresponding automorphic representation we expect to find will depend on that dimension.

If we drop the assumption that the Calabi-Yau manifold X is rigid, then h^3 will not be equal to two. It is still possible, nevertheless, that the Galois representation attached to the third cohomology contains an irreducible component of dimension two. If such a component occurs in $H^3(\overline{X}, \mathbb{Q}_\ell)$ for every ℓ and the resulting Galois representations are compatible, the same argument will apply. Such a system of compatible representations is usually described as a submotive of rank two.

Thus, exactly the same argument will show that any submotive of rank two is modular, i.e., the subrepresentation of dimension two over $\mathbb{Q}_\ell$ will be isomorphic to the ℓ-adic representation attached to a modular form f. The form f will be of weight 4 if the submotive is the $(3,0)+(0,3)$ part, and of weight 2 if the motive is contained in the $(2,1)+(1,2)$ part (and the representation will be a Tate twist of ρ_f).

There are several (proved and conjectural) examples of this in [8]. Many of them are of the type studied in [4], namely, Calabi-Yau threefolds containing a large number of elliptic ruled surfaces. To be specific, let

$$V_\ell = H^3(\overline{X}, \mathbb{Q}_\ell)^\vee.$$

The examples in [4] and [8] look like

$$V_\ell \cong V_\ell(f) \oplus [V_\ell(g_1) \otimes \chi_\ell] \oplus [V_\ell(g_2) \otimes \chi_\ell] \oplus \cdots \oplus [V_\ell(g_k) \otimes \chi_\ell],$$

where $h^3(X) = 2 + 2k$, f is a modular form of weight 4, the g_i are all modular forms of weight 2, and $V_\ell(h)$ is the ℓ-adic representation attached to a modular eigenform h. In many cases, Meyer finds examples where the g_i are in fact all the same; see, for example, pages 23–24 of [8].

4. Some speculations

Let X be a rigid Calabi-Yau threefold defined over $\mathbb{Q}$, and let f be the associated modular form of weight 4.

1) The level N of the form f is going to be a delicate arithmetic invariant of X (over $\mathbb{Q}$, rather than over the algebraic closure). The primes dividing N should be (exactly) the primes at which X has bad reduction in every model of X over $\mathbb{Z}$. The precise power of such primes that occurs in N presumably depends on the type of singularities, but we do not know how that should work.

2) Suppose we have X and its modular form f of level N. Then if we twist f by a quadratic character of level d, we get another eigenform of weight 4 and level dividing Nd^2. Will this form be attached to another rigid Calabi-Yau threefold over $\mathbb{Q}$? If so, then we should have $\overline{X} \cong \overline{X}_d$; in fact, the isomorphism should be defined over $\mathbb{Q}(\sqrt{d})$. In other words, this situation should correspond to the existence of rigid Calabi-Yau varieties that are not isomorphic over $\mathbb{Q}$ but are isomorphic over $\mathbb{C}$.

Helena Verrill has found an example where such X_d can be constructed (see [11]); does such a "twist" always exist? In [8], Meyer conjectured that the answer is yes.

3) Can we reverse this process? In other words, given an eigenform f of weight 4 on $\Gamma_0(N)$ and defined over $\mathbb{Q}$, does there exist a rigid Calabi-Yau threefold X corresponding to f? Since Barry Mazur first called attention to this question, it is known as *Mazur's problem.*

Of course, Mazur's problem is connected to the issue of whether there are infinitely many different birational equivalence classes of rigid Calabi-Yau threefolds. If the answer to Mazur's question is "yes," then we can translate the question to the setting of modular forms, where it becomes the question of understanding whether there are, up to twists, infinitely many modular forms of weight 4 (of any level) that are defined over $\mathbb{Q}$.

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