Non-compact Heegaard Splittings of Deleted Boundary 3-Manifolds:

"The eternal silence of these infinite spaces frightens me."

Blaise Pascal

Getting a handle(body) on non-compact 3-manifolds

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History

Theorem (Frohman-Meeks '97)**.** *Any two infinite-genus properly embedded one-ended complete minimal surfaces in* $\mathbb{R}^3$ *are properly ambient isotopic.*

- **Topological Aspect**
  Any two infinite genus Heegaard surfaces in $\mathbb{R}^3$ are properly ambient isotopic.

- **Geometric Aspect**
  Every one-ended complete minimal surface in $\mathbb{R}^3$ is a Heegaard surface.

Proper Ambient Isotopy

Definition. A proper ambient isotopy is map $f : M \times I \to M$ such that:

(i) $f_t : M \to M$ is an embedding for all t.

(ii) $f^{-1}(C) \subset M \times I$ is a compact set for all compact $C \subset M$.

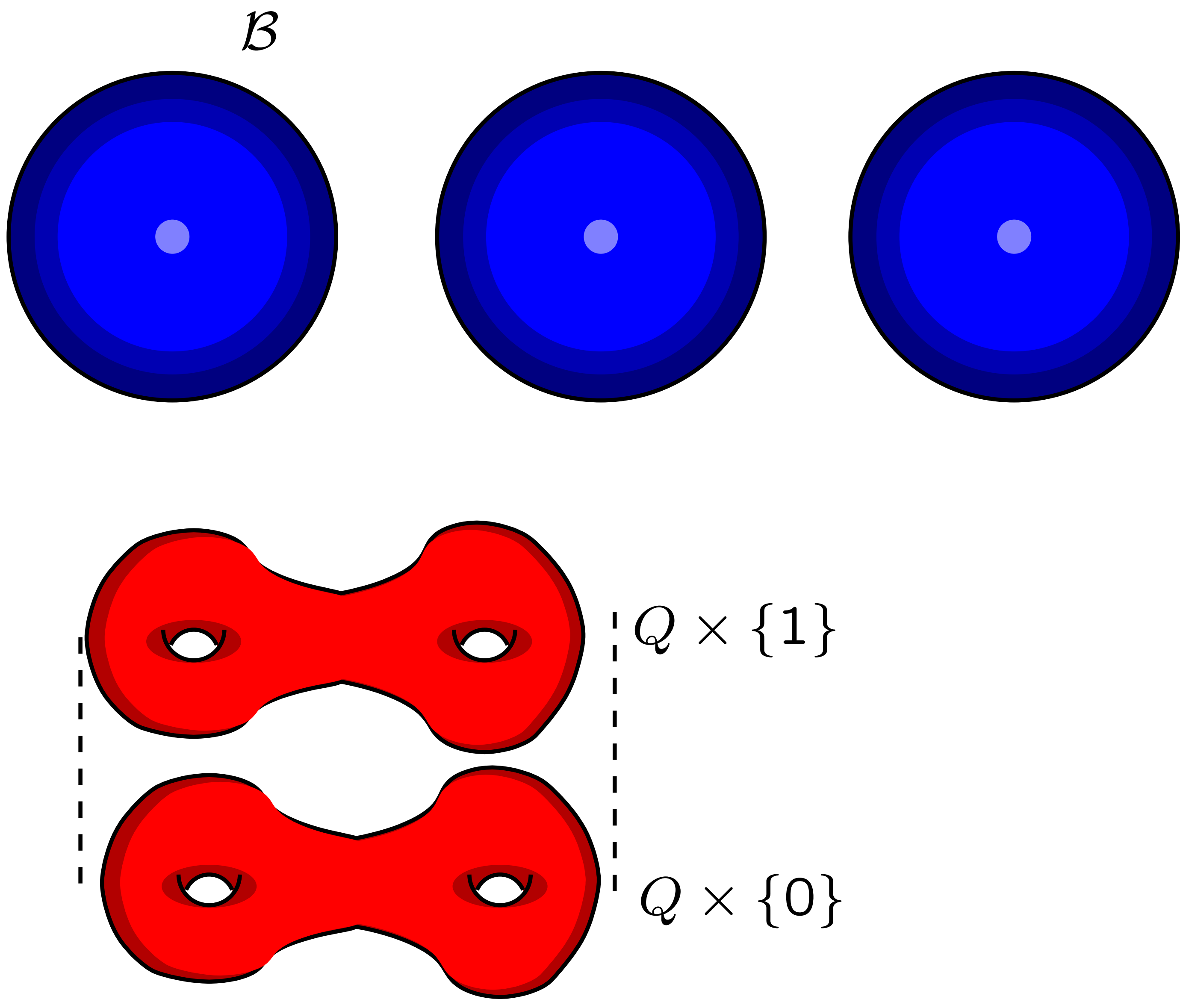
An Extension

Definition. Let $\overline{M}$ be a compact 3-manifold with $\partial\overline{M} \neq \varnothing$. If $M = \overline{M} - J$ where J is a collection of components of $\partial\overline{M}$ then M is a **deleted boundary manifold**.

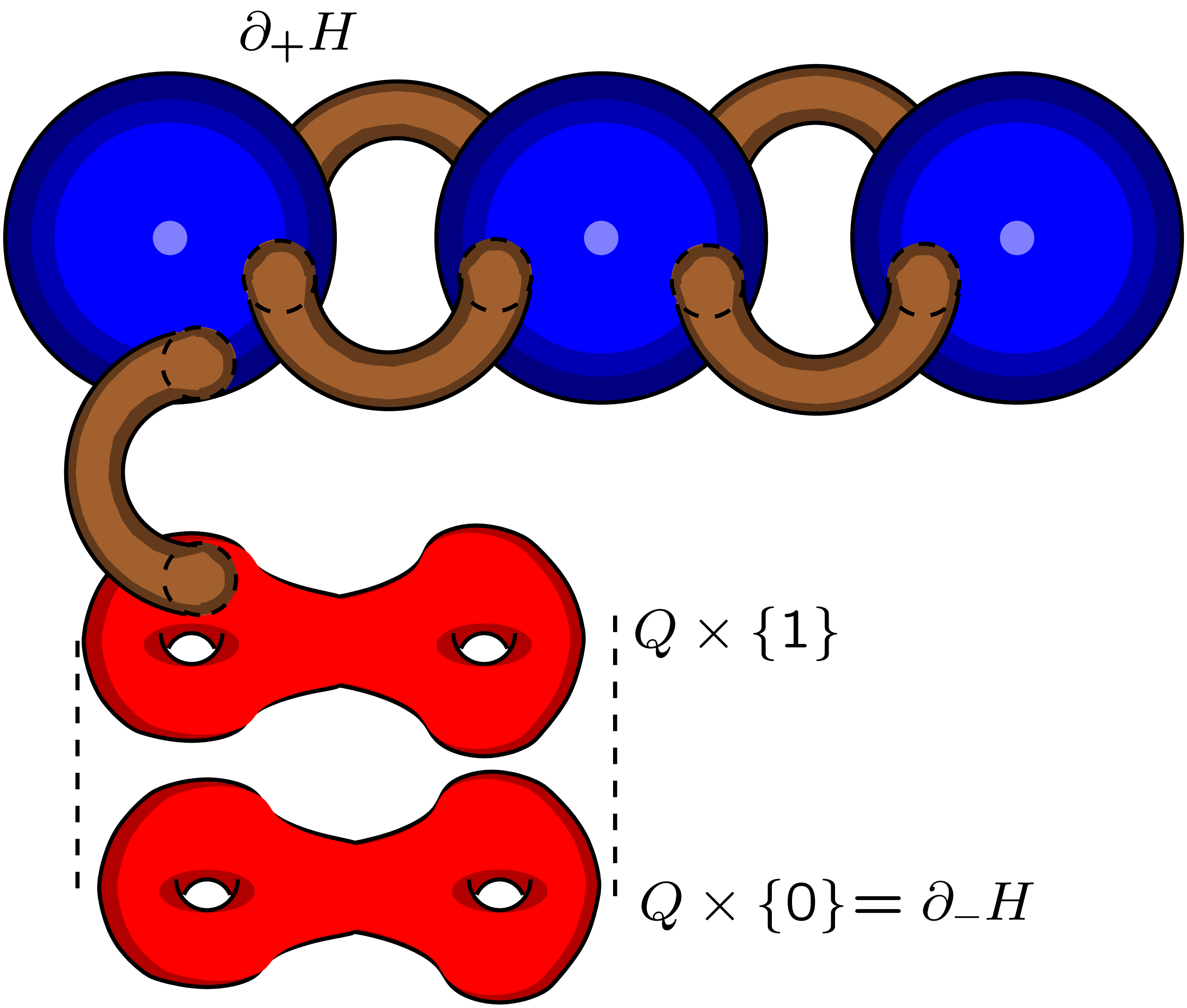
Example. Let F be a closed connected surface. Then $F \times \mathbb{R}$ is a deleted boundary 3-manifold.

Theorem (T). "Unique Heegaard Splittings"
If $M = \overline{M} - \partial\overline{M}$ is orientable and if $\partial\overline{M}$ contains no 2-spheres, then any two Heegaard splittings of M are properly ambient isotopic.

Building a Compressionbody



Building a Compressionbody



Definition of Compressionbody

- Q may be empty or disconnected or have boundary.

- Handles must be attached so that the result is locally finite.

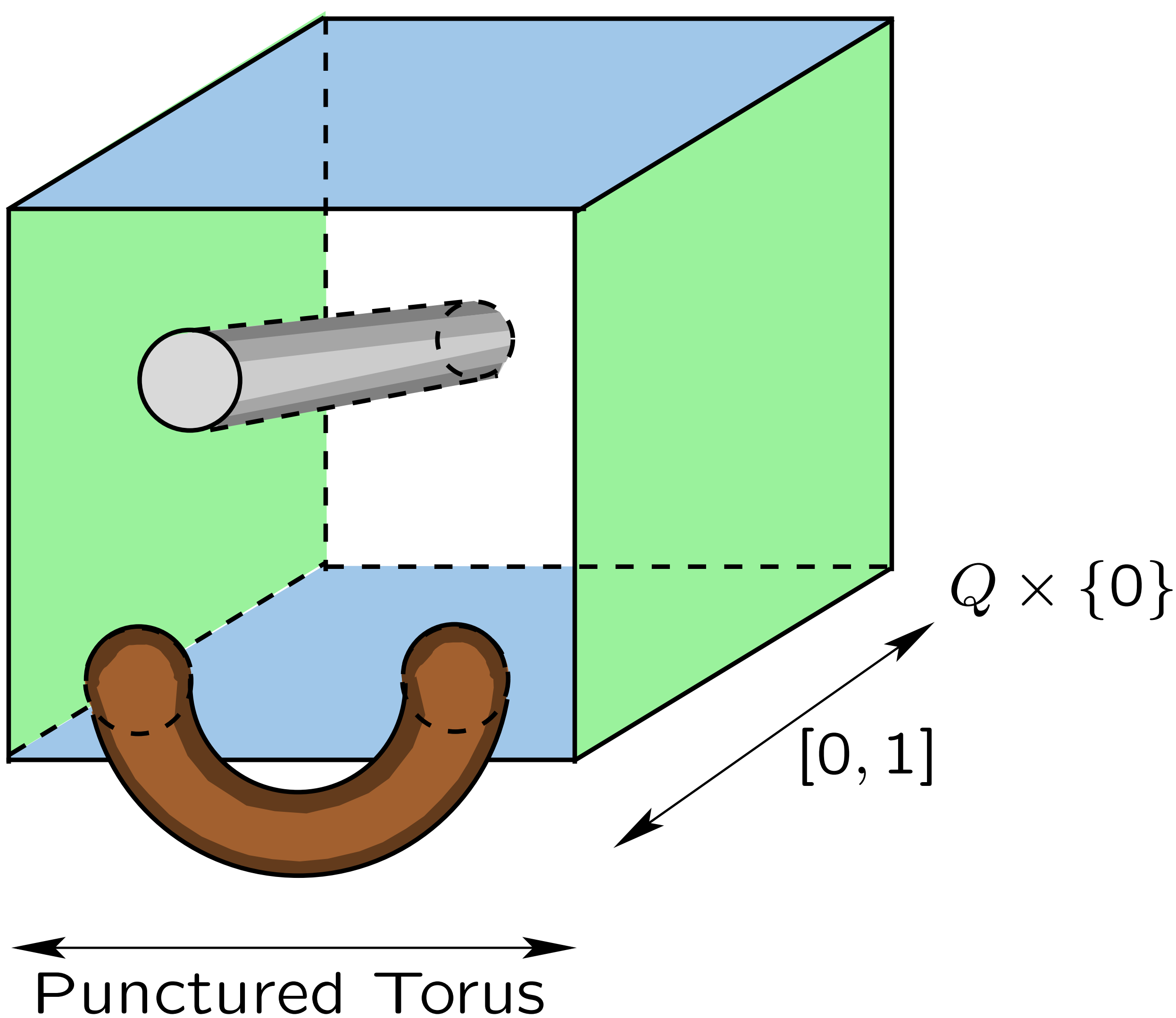
- If $Q = \varnothing$ then H is a **handlebody**.

- If $\partial Q = \varnothing$ then H is an **absolute compressionbody** otherwise H is a **relative compressionbody**.

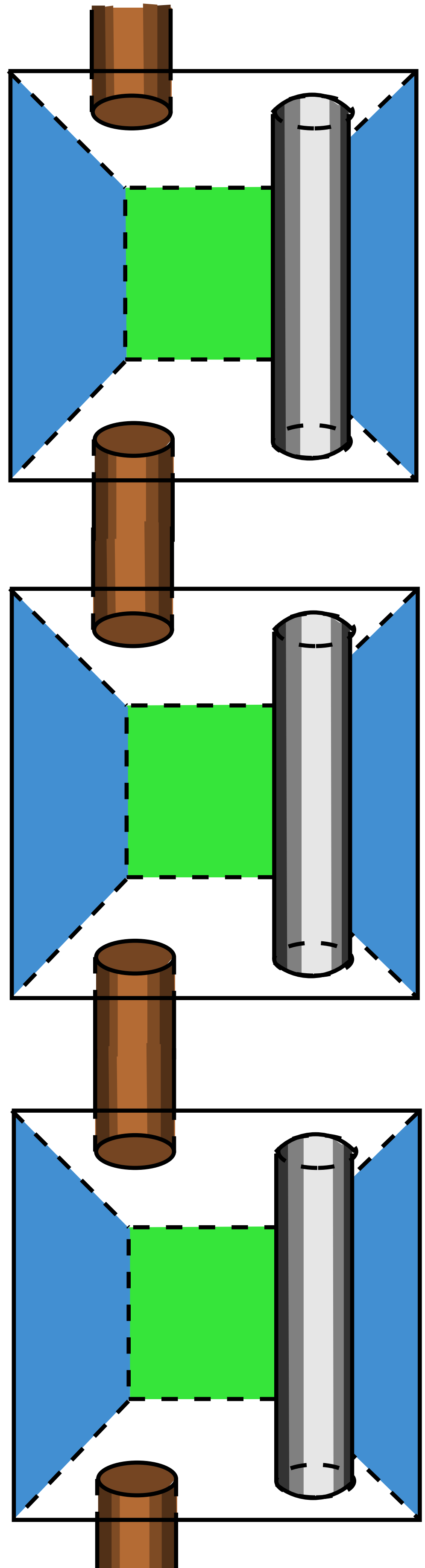
- $\partial_- H = Q \times \{0\}$.

- $\partial_+ H = \text{cl}(\partial H - \partial_- H)$ is the **preferred surface**.

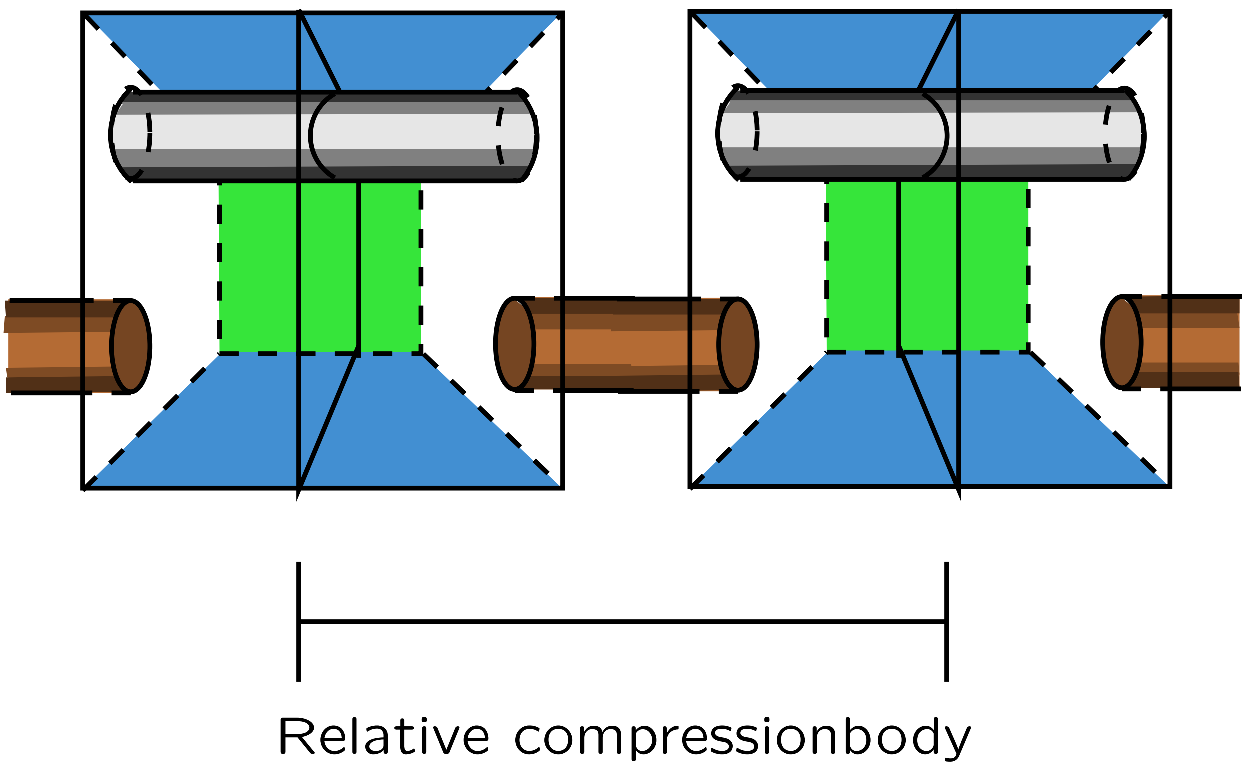
A relative compressionbody or a handlebody



An infinite genus handlebody



A relative compressionbody



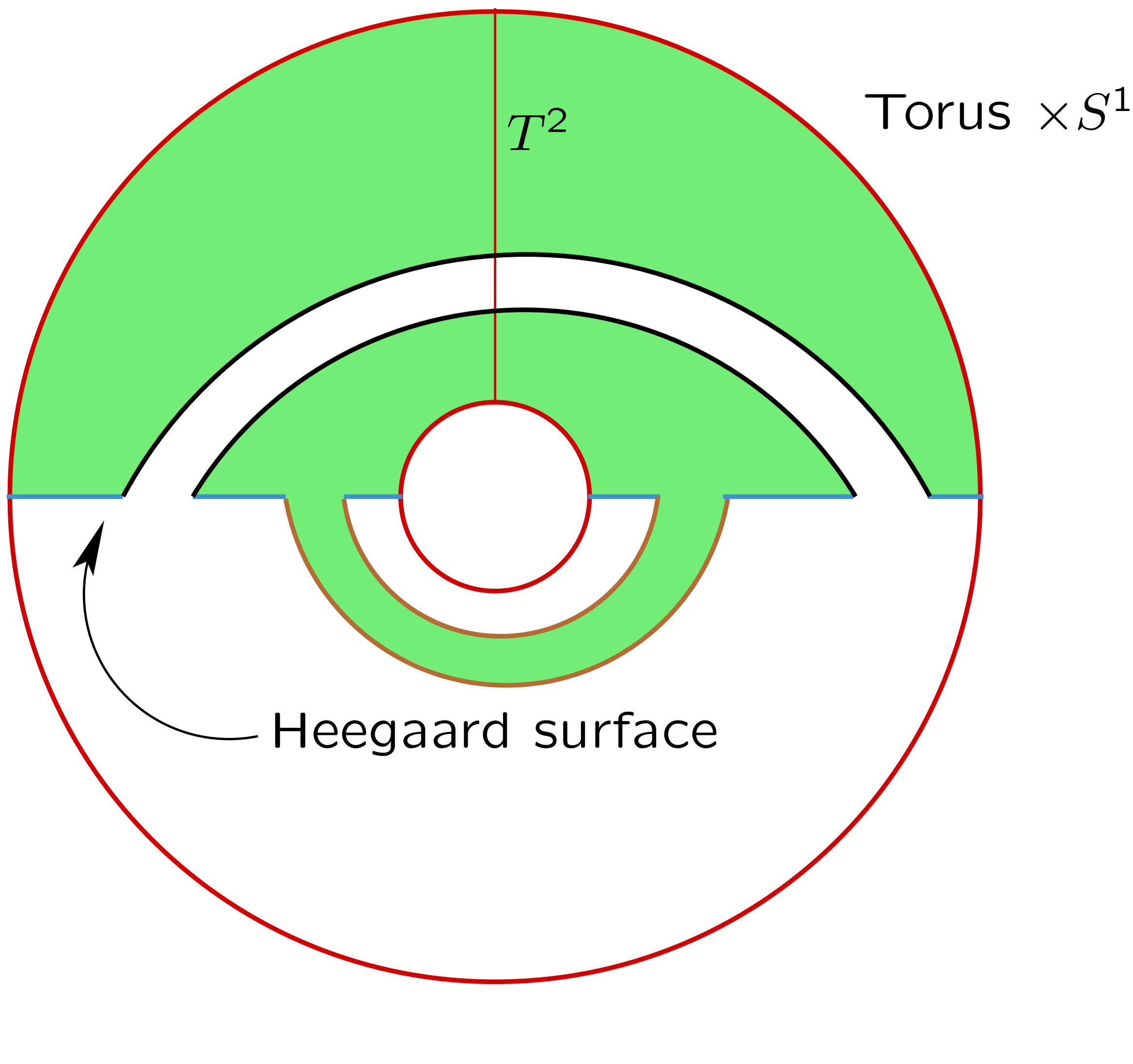
Heegaard Splittings

Definition. A **Heegaard splitting** is a decomposition $M = U \cup_S V$ into two compressionbodies glued along their preferred surfaces. If the compressionbodies are relative compressionbodies, the Heegaard splitting is a **relative Heegaard splitting**, otherwise it is an **absolute Heegaard splitting**.

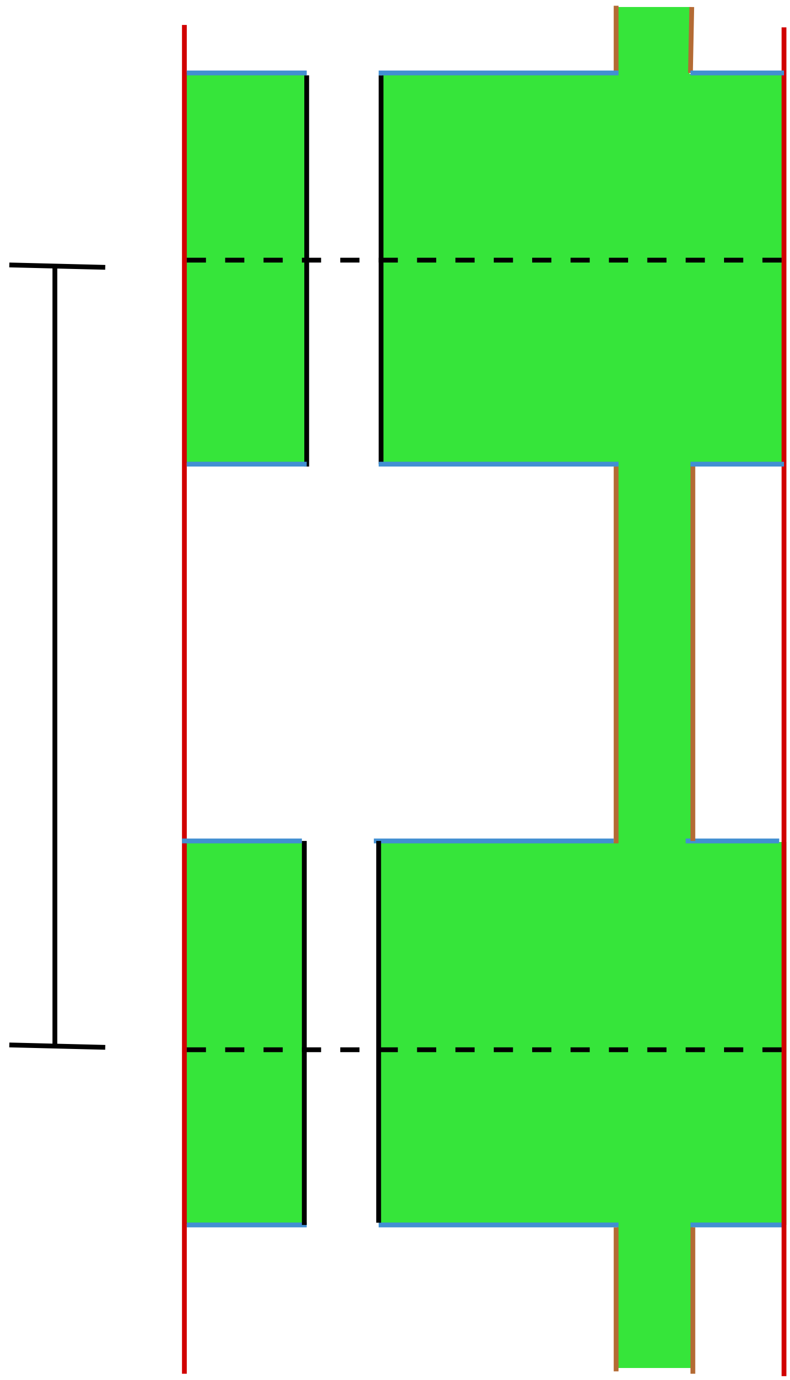
We will (usually) talk about absolute Heegaard splittings of non-compact 3-manifolds and relative Heegaard splittings of compact 3-manifolds.

A Heegaard splitting of $\mathbf{F} \times \mathbf{S}^1$

Let F be a closed connected orientable surface.



A Heegaard Splitting of $\mathbf{F} \times \mathbb{R}$



A relative

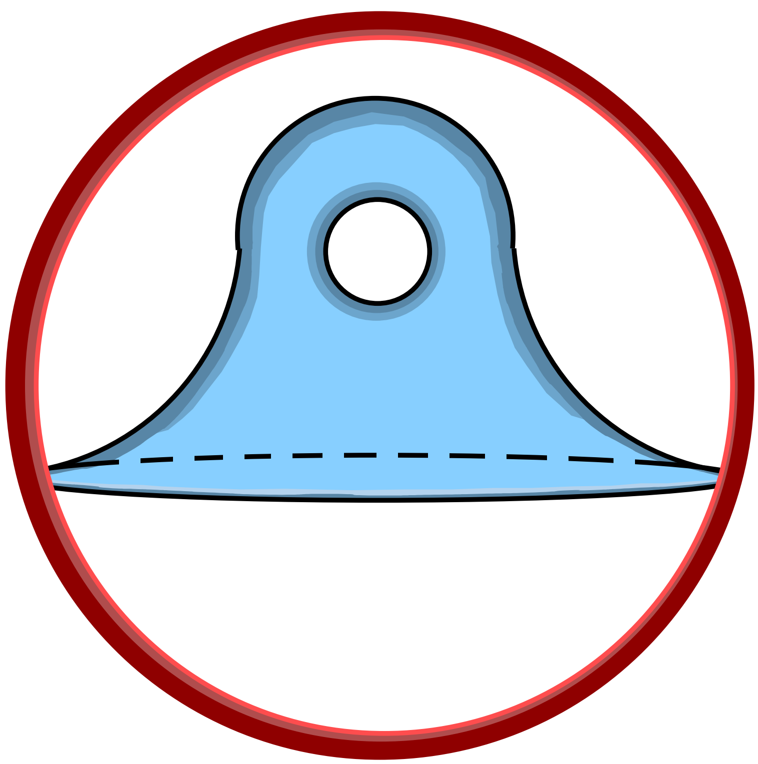
Heegaard splitting

Stabilizations

Definition. A Heegaard splitting

$$M = U \cup_S V$$

is **stabilized** if there is a ball in M which intersects S in a properly embedded unknotted once-punctured solid torus. The ball is called a **stabilizing ball**. The splitting is **end-stabilized** if for every compact set $C \subset M$ there is a stabilizing ball in each non-compact component of cl$(M - C)$.



Non-compact Analogues

Theorem (Frohman-Meeks)**.**
Any two end-stabilized Heegaard splittings of an open 3-manifold are properly ambient isotopic.

This is an analogue of the Reidemeister-Singer theorem which states that two (compact) Heegaard splittings are equivalent after finitely many stabilizations of each.

The Theorem

Recall that $M = \overline{M} - \partial\overline{M}$.

Theorem (T)**.** "Unique Heegaard Splittings"
If M is orientable and if $\partial\overline{M}$ contains no 2-spheres, then any two Heegaard splittings of M are properly ambient isotopic.

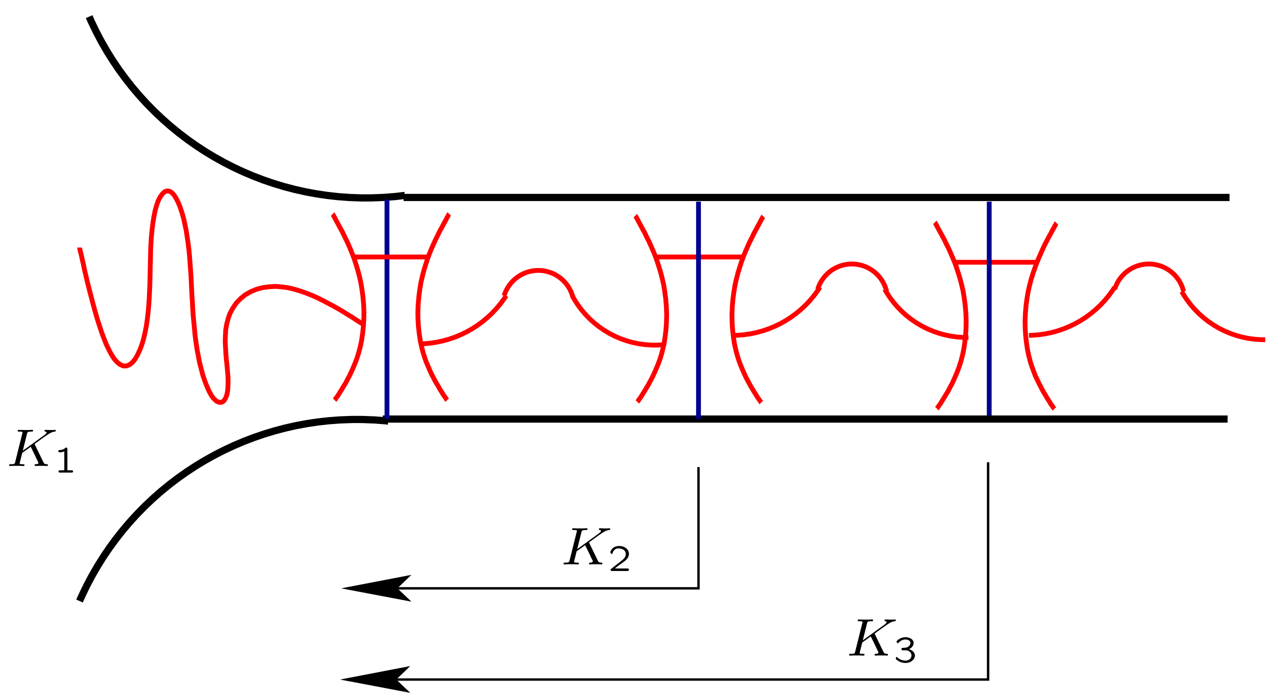
Goal: Show that every Heegaard splitting of M is end-stabilized.

The Idea

(1) Show that there is an exhausting sequence $\{K_i\}$ for M such that:

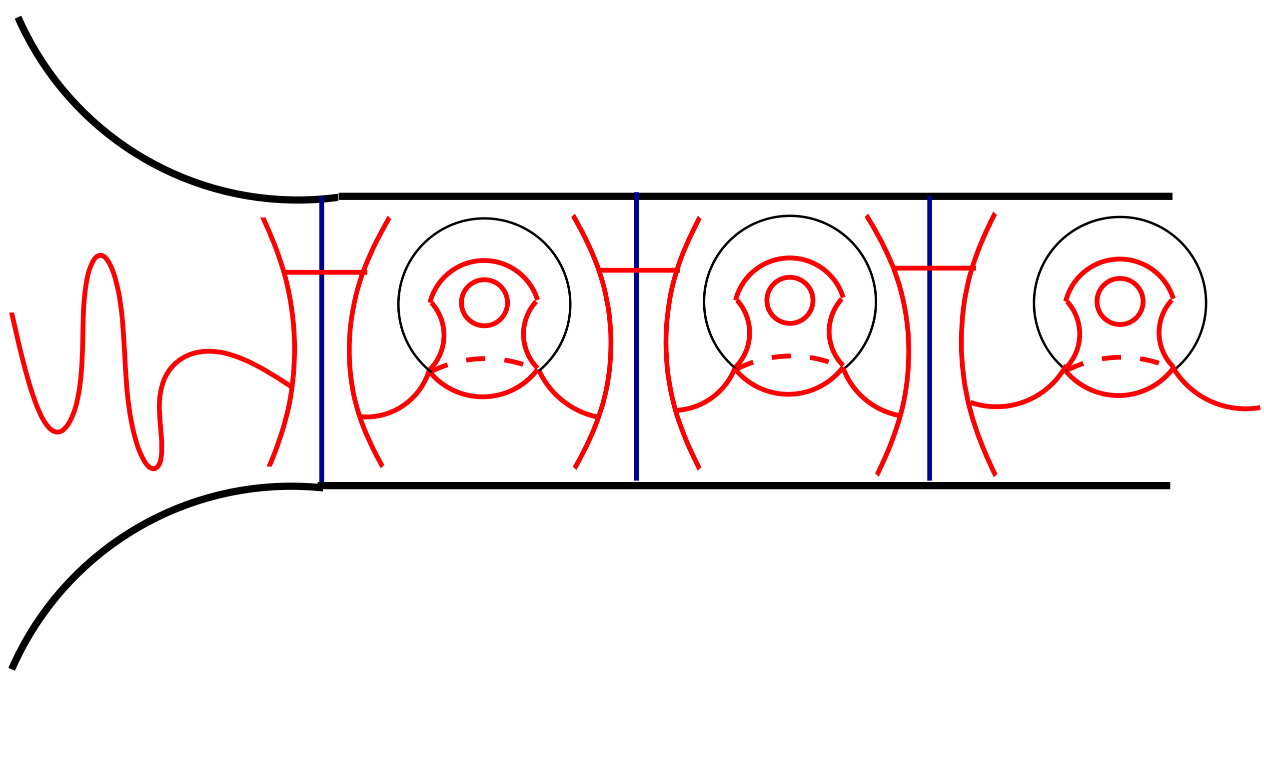
(i) $\mathrm{cl}(K_{i+1} - K_i)$ is homeomorphic to $\partial\overline{M} \times I$.

(ii) $S \cap \mathrm{cl}(K_{i+1} - K_i)$ is a relative Heegaard surface.



The Idea (continued)

(2) Apply Scharlemann and Thompson's classification of Heegaard splittings of $\partial\overline{M} \times I$ to conclude that (after passing to a subsequence of $\{K_i\}$) there is a stabilizing ball for S in each cl$(K_{i+1} - K_i)$. ■



Remarks on Step 1

- A (modified) version applies to a much larger class of non-compact 3-manifolds called "eventually end-irreducible 3-manifolds".

- It is a non-compact analogue of a theorem of Casson and Gordon which connects a certain type of Heegaard splitting with incompressible ($\approx \pi_1$-injective) surfaces.