ESSENTIAL SURFACES IN (3-MANIFOLD, GRAPH) PAIRS AND LEVELING EDGES OF HEEGAARD SPINES

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ABSTRACT. Let T be a graph in a compact, orientable 3–manifold M and let Γ be a subgraph. T can be placed in bridge position with respect to a Heegaard surface H. We show that if H is what we call (T,Γ)-c-weakly reducible in the complement of T then either a "degenerate" situation occurs or H can be untelescoped and consolidated into a collection of "thick surfaces" and "thin surfaces". The thin surfaces are essential in the graph complement and each thick surface is a strongly irreducible bridge surface in the complement of the thin surfaces. This extends previous results of Hayashi-Shimokawa and Tomova to graphs in 3-manifolds that may have non-empty boundary. We apply this result to the study of leveling edges of trivalent Heegaard spines.

1. INTRODUCTION

Thin position has been an important tool in knot theory and 3-manifold topology. Gabai [G] defined thin position for a knot in S^3 in his solution of the Poenaru Conjecture; it was also put to good use by Gordon and Luecke [GL] in their solution of the knot complement problem. Thin position was extended to graphs in S^3 by Scharlemann and Thompson [ST2] who used it to give a new proof of Waldhausen's classification of Heegaard splittings of S^3. Goda, Scharlemann, and Thompson [GST] also used thin position for graphs in S^3 to prove that an unknotting tunnel for a tunnel number one knot in minimal bridge position can be slid and isotoped to lie on the bridge surface. They used this fact to give a new proof of the classification of unknotting tunnels for 2–bridge knots. Goda, Scharlemann, and Thompson's result is also foundational to Cho and McCullough's [CM] recent important work on tunnel number one knots.

In tandem with their development of thin position for knots and graphs in S^3, Scharlemann and Thompson [ST1] described a very different thin position for 3–manifolds (based on earlier work of Casson and Gordon [CG]).

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In this type of thin position, a 3–manifold is decomposed along incompressible surfaces (called "thin surfaces") into codimension 0 submanifolds each of which contains a strongly irreducible Heegaard surface (called a "thick surface"). Thin position for a 3-manifold can be obtained by "untelescoping" a Heegaard surface for the 3–manifold into thin and thick surfaces. In practice, a strongly irreducible Heegaard surface functions very much like an incompressible surface. Part of the power of thin position for a 3-manifold comes from the fact that an irreducible Heegaard surface in a 3-manifold is either strongly irreducible, or can be untelescoped into a collection of incompressible surfaces and strongly irreducible surfaces. In either case, we have surfaces that function like incompressible surfaces. This aspect of thin position for 3–manifolds has been used, for example, in work on the virtually Haken conjecture [L, Ma].

Hayashi and Shimokawa [HS1] found a way of uniting (at least in spirit) the notions of thin position for a knot and thin position for a 3-manifold. Their version of thin position involves placing a properly embedded 1-manifold in a 3-manifold into bridge position with respect to a Heegaard surface for the 3-manifold and then untelescoping the Heegaard surface using compressing disks and bridge disks. Coward [C] used Hayashi and Shimokawa's thin position to find an algorithm for determining the bridge number of a hyperbolic knot in S^3.

Tomova [T1] strengthed Hayashi and Shimokawa's untelescoping operation by weakening the hypotheses on the sort of disks used in the untelescoping operation: she used so-called "c-disks". Tomova [T2] used this new version of thin position in her work on the relationship between multiple bridge surfaces for a given knot, producing a version of the "althernate Heegaard genus bounds distance" theorem of Scharlemann and Tomova [STo1]. Scharlemann and Tomova [STo2] also use this version of thin position to show that 2–bridge knots have essentially unique bridge surfaces. Tomova's version of thin position, however, requires that the 3-manifold be boundary-less.

In the present work, we show that (except in a few degenerate situations) a bridge surface for a graph in a 3-manifold, possibly with boundary, can be transformed using untelescoping-like operations into a type of thin position for the graph where the thin surfaces are essential in the graph complement and the thick surfaces are strongly irreducible bridge surfaces in the complement of the thin surfaces. The union of thin and thick surfaces is called a "multiple Heegaard splitting" for the graph in the 3-manifold.

In the thin position of Hayashi-Shimokawa and Tomova the proof that thin surfaces are essential (and not just incompressible) relies on the difficult classification of certain bridge surfaces [HS2, HS3]. So in our situation, proving that thin surfaces are essential relies on the classification of bridge surfaces for certain graphs in compression bodies [TT]. (See Theorem 3.5.)

Since in the past it has been helpful to be able to untelescope a bridge surface for a knot along c-disks (and not just compressing disks), we prove a relative version of the theorem. The set up is as follows: We have a compact orientable 3-manifold M, a Heegaard surface $H \subset M$ and a graph $T \subset M$ in bridge position with respect to H. Inside T we specify a subgraph Γ, and when we untelescope, we will use compressing disks for $H - T$ and so-called "cut disks" that intersect Γ. If M is a closed 3-manifold and T is a link then if Γ is chosen to equal T, we obtain Tomova's thin position. If M is a 3-manifold and T is a properly embedded 1-manifold, then choosing $\Gamma = \varnothing$ gives Hayashi and Shimokawa's thin position.

We prove

Theorem 7.2 (Rephrased). *Let M be a compact, orientable 3-manifold containing a properly embedded graph T and let Γ be a subgraph of T. Assume that $M - T$ is irreducible and that no sphere in M intersects T exactly once. Let H be a Heegaard surface for M and suppose that T is in bridge position with respect to H. Then one of the following holds:*

(1) *H is Γ-c-strongly irreducible*
(2) *there is a multiple Γ-Heegaard splitting $\mathscr{H}$ for (M,T,Γ) so that each thin surface is T–essential and each thick surface is (T,Γ)-c-strongly irreducible in the component of $M - \mathscr{H}^-$ containing it. Furthermore, $\mathscr{H}$ is obtained by applying "untelescoping-like" operations to H.*
(3) *H contains a generalized stabilization,*
(4) *H is a perturbed bridge surface,*
(5) *T has a removable path.*

If H contains a generalized stabilization, it can be modified to become a simpler bridge surface for T. If T has a removable path, then a subgraph of T is essentially a "core" of one of the compression bodies in the complement of H. This subgraph can be drilled out making H into a simpler bridge surface for a new graph in a new 3-manifold. If H is perturbed, either it can be unperturbed, once again simplifying the bridge presentation of T, or an edge (called a "perturbed level edge" or "perturbed level cycle") of T can be isotoped onto H. This gives a way of possibly extending some of Goda-Scharlemann-Thompson's tunnel leveling results to knots other than tunnel number one knots in S^3. As first steps in this direction, we give the following application:

Theorem 9.1. *Suppose H is a Heegaard surface for M and T is an irreducible trivalent Heegaard spine for a closed manifold M in minimal bridge position with respect to H. Then one of the following occurs:*

1. *H is stabilized, meridionally stabilized, or bimeridionally stabilized as a splitting of (M,T).*
2. *T has a perturbed level edge.*
3. *T contains a perturbed level cycle.*
4. *There is an essential meridional surface F in the exterior of T such that* $\text{genus}(F) \leq \text{genus}(H)$.

2. Definitions

2.1. **Surfaces in (M,T).** Let T be a finite graph. Unless otherwise specified we assume that T has no valence 2 vertices as such vertices can generally be deleted and their adjacent edges amalgamated. We say that T is *properly embedded* in a 3–manifold M if $T \cap \partial M$ is the set of all valence 1 vertices of T. We will denote the pair (M,T).

Suppose that $F \subset M$ is a surface such that $\partial F \subset (\partial M \cup T)$. Then F is *T–compressible* if there exists a compressing disk for $F - T$ in $M - T$. If F is not T–compressible, it is *T–incompressible*. F is *T–∂–compressible* if there exists a disk $D \subset M - T$ with interior disjoint from F such that ∂D is the endpoint union of an arc γ in F and an arc δ in ∂M. We require that γ not be parallel in $F - T$ to an arc of $\partial F - T$. If F is not T–∂–compressible, it is *T–∂–incompressible*. Finally suppose Γ is some subgraph of T. We will say that F is *(T,Γ)-cut-compressible*, if there exists a compressing disk D^c for $F - T$ in M so that $|D^c \cap T| = 1$ and that point is contained in Γ. We also require that ∂D^c is not parallel in $F - T$ to a puncture $T \cap F$. We call D^c a *(T,Γ)-cut-disk*. A *(T,Γ)-c-disk* will be either a T-compressing disk or a (T,Γ)-cut-disk. A surface F in M is called *T-parallel* if F is boundary parallel in $M - \mathring{\eta}(T)$ and *T-essential* if it is T-incompressible and not T-parallel.

2.2. **Trivially embedded graphs in compression bodies.** Let C be a compression body and T be a properly embedded graph in C. A connected component τ of T is *trivial* in C if it is one of four types:

1. *Bridge arc:* a single edge with both endpoints in $\partial_+ C$ which is parallel to an arc in $\partial_+ C$. The disk of parallelism is called a *bridge disk*.
2. *Vertical edge:* a single edge with one endpoint in $\partial_+ C$ and one endpoint in $\partial_- C$ that is isotopic to $\{\text{point}\} \times I$.
3. *Pod:* a graph with a single vertex in the interior of C and with all valence 1 vertices lying in $\partial_+ C$ so that there is a disk D with $\partial D \subset$

$\partial_+ C$ inessential in $\partial_+ C$ and so that $\tau \subset D$. The disk D is called a *pod disk*. Each of the components of $D - \tau$ will be called a *bridge disk* as these components play the same role as bridge disks for bridge arcs.

(4) *Vertical pod:* a graph with a single vertex in the interior of C and with one valence 1 vertex lying in $\partial_- C$ and all other valence 1 vertices in $\partial_+ C$ so that if the edge adjacent to $\partial_- C$ is removed the resulting graph is a pod and if instead all but one of the edges adjacent to $\partial_+ C$ are removed, the result is a vertical edge with a valence 2 vertex in its interior. The edges that have one endpoint in $\partial_+ C$ are called *pod legs* and the other edge is called a *pod handle*.

If all components of T are trivially embedded, then we say that T is *trivially embedded* in C.

2.3. Trivially embedded graphs in Γ-compression bodies.

Definition 2.1. *Let C be a compression body containing a properly embedded graph T and let Γ be a subgraph of T. Suppose that there is a collection of (T,Γ)-cut-disks, $\mathscr{D}^c$, such that:*

(1) *for each edge of Γ there is at most one disk in $\mathscr{D}^c$ intersecting it,*
(2) *each edge of Γ intersected by $\mathscr{D}^c$ has both endpoints on $\partial_- C$,*
(3) *cut-compressing C along all cut-disks $\mathscr{D}^c$ produces a union of compression bodies $C_1,..,C_n$,*
(4) *for each i the graph $C_i \cap T$ is trivially embedded in C_i.*

Then we call the triple of the compression body and the graphs a Γ-compression body containing a trivially embedded graph $T - \Gamma$ *and denote it* (C,T,Γ).

Note that if $\Gamma = \emptyset$ then $(C,T,\Gamma) = (C,T)$ is a compression body containing a trivially embedded graph.

2.4. Heegaard surfaces and Γ-Heegaard surfaces.

Let (M,T) be a compact connected orientable 3–manifold containing a properly embedded graph. A *Heegaard splitting* for (M,T) is a decomposition of M into two compression bodies, C_1 and C_2, such that $T_i = T \cap C_i$ is trivially embedded in C_i for $i \in \{1,2\}$. The surface $H = \partial_+ C_1 = \partial_+ C_2$ is called a *Heegaard surface* for (M,T). We will also say that T is in *bridge position* with respect to H and that H is a *bridge splitting* of (M,T).

Suppose now that Γ is a subgraph of T and H is a surface in (M,T) transverse to T so that H splits M into compression bodies C_1 and C_2 such that (C_i, T_i, Γ_i) is a Γ_i-compression body containing a trivially embedded graph $T_i - \Gamma_i$ for $i \in \{1,2\}$. In this case we say H is Γ-Heegaard surface or a Γ-bridge surface for (M,T,Γ). If $\Gamma = \emptyset$ then H is simply a Heegaard surface.

Suppose H is a Γ-Heegaard surface for (M,T,Γ) splitting it into triples (C_1,T_1,Γ_1) and (C_2,T_2,Γ_2). We will say that H is *T-reducible* if there exists a sphere S disjoint from T such that $S \cap H$ is a single curve essential in $H - \eta(T)$, otherwise H is *T-irreducible*. We will say that H is *T-weakly reducible* if H has T-compressing disks on opposite sides with disjoint boundaries. Otherwise H is said to be *T-strongly irreducible*. We will say that H is *(T,Γ)-c-weakly reducible* if H has (T,Γ)-c-disks on opposite sides with disjoint boundaries. Otherwise H is said to be *(T,Γ)-c-strongly irreducible*.

2.5. **Generalized stabilizations, perturbations and removable paths.** Several geometric operations can be used to produce new Γ-bridge surfaces from old ones. These are generalizations of stabilizations for Heegaard splittings of manifolds and usually we work with bridge surfaces that are not obtained from others via these operations. A more detailed discussion of these operations can be found in [HS2, STo2, TT]. A Γ-bridge surface H for (M,T,Γ) will be called *stabilized* if there is a pair of T-compressing disks on opposite sides of H that intersect in a single point. The Γ-bridge surface is *meridionally stabilized* if there is a (T,Γ)-cut-disk and a T-compressing disk on opposite sides of H that intersect in a single point. H will be called *bimeridionally stabilized* if there are two (T,Γ)-cut-disks on opposite sides of H that intersect in a single point. The concept of "bimeridionally stabilized" is not used in the main theorem; it shows up only in the application.

As we are considering manifolds with boundary there are two other geometric operations that can be used to obtain a new bridge surface from an old one. Suppose H is a Γ-Heegaard splitting for (M,T,Γ) decomposing M into compression bodies C_1 and C_2. Let F be a component of $\partial_- C_1 \subset \partial M$ and let T' be a collection of vertical edges in $F \times [-1,0]$ so that $T' \cap (F \times \{0\}) = T \cap F$. Let H' be a minimal genus Heegaard surface for $(F \times [-1,0], T')$ which does not separate $F \times \{-1\}$ and $F \times \{0\}$ and which intersects each edge in T' exactly twice. H' can be formed by tubing two parallel copies of F along a vertical arc not in T'. We can form a Γ-Heegaard surface H'' for $M \cup (F \times [-1,0])$ by *amalgamating* H and H'. This is simply the usual notion of amalgamation of Heegaard splittings (see [Sc]). In fact, H'' is a Γ-Heegaard surface for $(M \cup (F \times [-1,0]), T \cup T')$. Since $(M \cup (F \times [-1,0]), T \cup T')$ is homeomorphic to (M,T), we may consider H'' to be a Γ-Heegaard surface for (M,T,Γ). H'' is called a *boundary stabilization* of H. A similar construction can be used to obtain a new Γ-Heegaard splitting of (M,T,Γ) by tubing two parallel copies of F along a vertical arc that does lie in $T' \subset \Gamma$. In this case H'' will be called *meridionally boundary stabilized*.

If a Γ-bridge surface is stabilized, boundary stabilized, meridionally stabilized, or meridionally boundary stabilized we will say that it contains a *generalized stabilization*.

A Γ-bridge surface is called *cancellable* if there is a pair of bridge disks D_i on opposite sides of H such that $\emptyset \neq (\partial D_1 \cap \partial D_2) \subset (H \cap T)$. If $|\partial D_1 \cap \partial D_2| = 1$ we will call the bridge surface *perturbed*. Unlike the case when T is a 1-manifold, a perturbed bridge surface cannot necessarily be unperturbed by an isotopy. If neither of ∂D_i contains a vertex of T, then a new bridge surface H' for (M,T) can be found so that $|H' \cap T| < |H \cap T|$. If both ∂D_1 and ∂D_2 contain a vertex of T then a simpler bridge splitting does not necessarily exist. However, in this situation T often has a "perturbed level edge" (see below). Lemmas 9.5 and 9.6 provide more details on when perturbed bridge splittings can be unperturbed.

We say that T has a *perturbed level cycle* σ if there is a pair of bridge disks $\{D_1, D_2\}$ on opposite sides of H such that $(\partial D_1 \cap \partial D_2) \subset (T \cap H)$, and $\sigma \subset \partial(D_1 \cap \partial D_2)$. T has a *perturbed level edge* if T is perturbed by disks D_1, D_2 such that both ∂D_1 and ∂D_2 contain an interior vertex of T. The pair $\{D_1, D_2\}$ is called the *associated cancelling pair of disks*.

Suppose that $\zeta \subset T$ is a 1–manifold which is the the union of edges in T (possibly a closed loop with zero or more vertices of T). We say that ζ is a *removable path* if the following hold:

(1) Either the endpoints of ζ lie in ∂M or ζ is a cycle in T.
(2) ζ intersects H exactly twice
(3) If ζ is a cycle, there exists a cancelling pair of disks $\{D_1, D_2\}$ for ζ with $D_j \subset C_j$. Furthermore there exists a compressing disk E for H such that $|E \cap T| = 1$ and if $E \subset C_j$ then $|\partial E \cap \partial D_{j+1}| = 1$ (indices run mod 2) and E is otherwise disjoint from a complete collection of bridge disks for $T - H$ containing $D_1 \cup D_2$.
(4) If the endpoints of ζ lie in ∂M, there exists a bridge disk D for the bridge arc component of $\zeta - H$ such that $D - T$ is disjoint from a complete collection of bridge disks Δ for $T - H$. Furthermore, there exists a compressing disk E for H on the opposite side of H from T such that $|E \cap D| = 1$ and E is disjoint from Δ.

If T has a removable path ζ, ζ can be isotoped to lie in a spine for one of the compression bodies C_1 or C_2.

2.6. **Γ-c-Heegaard surfaces and removable edges.** This section presents a technical result which sometimes allows a Γ-c-Heegaard surface to be converted into a Heegaard surface with removable edges.

Suppose that H is a Γ-Heegaard surface for (M, T, Γ) and that $e \subset \Gamma$ is an edge disjoint from H with both endpoints in ∂M. Let E be a cut disk intersecting e whose boundary is in H. By isotoping e so that $e \cap E$ moves through ∂E to the other side, we convert e into a removable path e'. Let T' be the new graph. Let H' be the new Γ-Heegaard surface for (M, T'). Let D be the bridge disk for the bridge arc component of $e' - H'$.

Lemma 2.2. *If H' is stabilized or meridionally stabilized then so is H. If H' is boundary-stabilized or meridionally boundary stabilized, so is H and the stabilization is along the same component of ∂M. If H' is perturbed then so is H. If T' contains a removable path other than e' then either that path is removable in T or H is meridionally stabilized.*

Proof. **Case 1:** Suppose that H' is stabilized or meridionally stabilized by disks D_1 and D_2 on opposite sides of H' which intersect once. Out of all such pairs of stabilizing or meridionally stabilizing pairs, choose D_1 and D_2 so that $|(D_1 \cup D_2) \cap (D \cup E)|$ is minimal. The next claim shows that (D_1, D_2) also stabilizes or meridionally stabilizes H.

Claim: $|(D_1 \cup D_2) \cap e| \leq 1$.

Since (D_1, D_2) (meridionally) stabilize H', $|(D_1 \cup D_2) \cap e'| \leq 1$. Without loss of generality, choose the labelling so that D_1 is on the same side of H as D. Then $D_1 \cap e = \varnothing$, and so $(D_1 \cup D_2) \cap e = D_2 \cap e$. If D_2 is disjoint from and not parallel to E, then clearly $|e \cap D_2| \leq 1$. If D_2 is parallel to E, then $|D_2 \cap e| = |E \cap e| = 1$. If D_2 is not disjoint from E, then by the minimality of $|(D_1 \cup D_2) \cap (E \cup D)|$ it follows that $D_2 \cap E$ is a collection of arcs. After possibly a small isotopy, we may assume that $D \cap E$ is disjoint from $D_2 \cap E$. Then using D to isotope e' back to e guarantees that e is disjoint from D_2. □(Claim)

Case 2: Suppose that H' is boundary stabilized or meridionally boundary stabilized. In this case there exists a (T, Γ)-c-disk D' for H such that compressing H' along D' produces surfaces H_1 and H_2 and the surface H_2 bounds a product region with ∂M containing only vertical arcs of T while the surface H_1 is a Γ-Heegaard surface for (M, T, Γ).

If D' is disjoint from or parallel to E then it is clear that H is boundary stabilized or meridionally boundary stabilized. Suppose, therefore that D' intersects E. We may assume that D' was chosen so as to minimize $D' \cap E$. This implies that $D' \cap E$ is a non-empty collection of arcs. Then, as in Case 1, isotoping e' back to e shows that $e \cap D' = \varnothing$. Hence, H is boundary stabilized or meridionally boundary stabilized and the stabilization is along the same component as that of H'.

Case 3: Suppose that (D_1, D_2) are a perturbing pair of disks for H' such that D_1 is on the same side of H as D. Notice that $\partial D_1 \cup \partial D_2$ is disjoint from e' since two components of $e' - H'$ are vertical arcs in the compression body containing them. Thus, unless $e \cap D_2 \neq \varnothing$, (D_1, D_2) is a perturbing pair for H. If $D_2 \cap e \neq \varnothing$, then D_2 intersects the neighborhood of E used to push e to e'. An argument similar to that of Cases 1 and 2 shows that ∂D

can be assumed to be disjoint from ∂D_2, and so e is disjoint from D_2, as desired.

Case 4: Suppose that T' contains a removable path $\zeta \neq e$. An argument similar to the previous cases shows that ζ is a removable path in T, unless ζ is not a cycle and the compressing disk E from condition (4) of the definition of removable path is equal to the present disk E. Suppose, therefore, that this is the case. By the definition of removable path, there is a bridge disk D' for $e' - H'$ which is disjoint from E. A small isotopy of $D \cup D'$ creates a compressing disk E' for $H' - T'$ intersected once by E. The pair (E, E') therefore, shows that H' is stabilized. Since $\partial E'$ is non-separating on H' and therefore on H, E' is a compressing disk for H disjoint from T. The boundary of E' intersects E exactly once and $E \cap T = E \cap e$ and so H is meridionally stabilized. □

2.7. **Multiple bridge splittings.** To prove our main theorem we will extend to graphs the definition of multiple bridge splittings for the pair (3-manifold, 1-manifold) introduced by Hayashi and Shimokawa in [HS3] and generalized by Tomova in [T1].

Definition 2.3. *Suppose M is a 3-manifold containing a properly embedded graph T and let Γ be a subgraph of T. A disjoint union of surfaces $\mathcal{H}$ is a multiple Γ-Heegaard splitting for (M,T,Γ) if:*

- *the closure of each component of $(M,T) - \mathcal{H}$ is a Γ-compression body C_i containing a trivially embedded graph $T_i = (T - \Gamma) \cap C_i$,*
- *for each i, $\partial_+ C_i$ is attached to some $\partial_+ C_j$ and $\partial_- C_i$ is either contained in ∂M or is attached to some $\partial_- C_k$.*
  Let $\mathcal{H}^+ = \cup \partial_+ C_i$ and $\mathcal{H}^- = \cup \partial_- C_i$

3. Properties of compression bodies containing properly embedded graphs

In this section we will generalize many of the well known results for compression bodies to the case when the compression body contains a graph embedded in a specific way.

Lemma 3.1. *Suppose C is a Γ-compression body containing a trivially embedded graph $T - \Gamma$. Compressing or cut-compressing C results in a union of Γ-compression bodies each containing a trivially embedded graph.*

Proof. Given any (T,Γ)-c-disk D^c for (C,T,Γ) we can always find a collection of pairwise disjoint (T,Γ)-c-disks $\mathcal{D}^c$ containing D^c so that (C,T,Γ) c-compressed along $\mathcal{D}^c$ is a collection 3-balls and components homeomorphic to $G \times I$ where G is a component of $\partial_- C$. Both types of components may contain trivially embedded graphs. The result follows. □

The next lemma will imply that the negative boundary of a Γ–compression body is incompressible in the complement of T.

Lemma 3.2. *Suppose (C,T,Γ) is a Γ-compression body containing a trivially embedded graph $T-\Gamma$. If F is a (T,Γ)-c-incompressible, T-∂-incompressible surface in C transverse to T, then F is a collection of the following kinds of components:*

- *spheres that bound balls in C: each ball may intersect T in at most one component and this component can include at most one vertex of T,*
- *disks intersecting T in 0 or 1 points,*
- *vertical annuli disjoint from T,*
- *closed surfaces parallel to components of $\partial_- C$ so that the region of parallelism intersects the graph in vertical arcs.*

Proof. Let $\mathscr{D}^c$ be the collection of (T,Γ)-c-disks so that C compressed along $\mathscr{D}^c$ is a union of pairs (C_i,T_i) where C_i is a compression body and T_i is a graph trivially embedded in C_i. As F is (T,Γ)-c-incompressible and T-∂-incompressible, it can be isotoped to be disjoint from $\mathscr{D}^c$. Without loss of generality suppose $F \subset C_1$.

Suppose τ is a component of T_1 which is a pod or a vertical pod. Let D be a pod disk for τ chosen so that $|F \cap D|$ is minimal. Because F is incompressible and ∂-incompressible F cannot intersect D in arcs or simple closed curves disjoint from τ. In fact if F intersects D at all, F must be either a once punctured disk intersecting the pod disk in an arc α so that $|\alpha \cap l| = 1$ where l is a pod leg of τ, a twice punctured sphere that intersects the pod disk in a simple closed curve β so that $\beta \cap l = 2$, or a sphere that bounds a ball containing the interior vertex of τ. In the last case $F \cap D$ is a single simple closed curve that is the boundary of a neighborhood of the vertex.

We conclude that either F is one of the first three types of components or it is disjoint from all pod-disks. In the latter case F is contained in one of the components obtained from C_1 by removing all pod-disks. Therefore F is either contained in a ball disjoint from T or in the compression body (C',T') where C' is isotopic to C_1 and T' is a trivially embedded tangle in C'. In either case the desired result follows by [HS2, Lemma 2.4]. □

Corollary 3.3. *If (C,T,Γ) is a Γ-compression body containing a trivially embedded graph $T-\Gamma$, then $\partial_- C$ is T-incompressible.*

Lemma 3.4. *Suppose (C,T,Γ) is a Γ-compression body containing a trivially embedded graph $T-\Gamma$. If F is a connected T-incompressible surface such that $F \cap \partial C = \emptyset$, then F bounds a compression body C_F in C so that $\partial_- C_F \subset \partial_- C$ and F is T-parallel to $\partial_- C_F$.*

Proof. If F is a (T,Γ)-c-incompressible surface, the result follows from Lemma 3.2 so suppose this is not the case. Maximally (T,Γ)-cut-compress F and note that (T,Γ)-c-compressing never creates T-compressing disks for the surface. Therefore, by Lemma 3.2 the resulting collection of (T,Γ)-c-incompressible surfaces $\mathscr{F}$ consists of two kinds of components: spheres that bound balls in C so that each ball intersects T in at most one component and this component contains at most one vertex of T, and closed surfaces parallel to components of $\partial_- C$ so that the regions of parallelism intersect the graph in vertical arcs. Furthermore note that each component of $\mathscr{F}$ intersects Γ in at least one point so there are no sphere components disjoint from T.

Suppose $\mathscr{F}$ contains a sphere component S' which bounds a ball B that intersects T in a single edge. As cut-disks have boundaries that are essential curves in $F-T$, such a sphere can only be the result of cut-compressing a torus which is the boundary of a regular neighborhood of a closed component of T with no vertices. As this torus is disjoint from T, it is in fact the original surface F and the result follows. Therefore from now on we may assume that any sphere components of $\mathscr{F}$ bound balls that contain a vertex of T.

The surface F can be recovered from $\mathscr{F}$ by adding to $\mathscr{F}$ a collection of tunnels Λ that run along edges of Γ and are dual to the cut-disks. Let F' be an outermost component that is T-parallel to a component of $\partial_- C$, i.e., F' is not contained in the region of parallelism of any other component. Let P' be the region of parallelism between F' and some component of $\partial_- C$. By Lemma 3.2 there are no vertices of T in P' and therefore none of the sphere components of $\mathscr{F}$ are contained in P'. Suppose F'' is a component of $\mathscr{F}$ contained in P' and adjacent to F'. As F is connected there must be $\lambda \in \Lambda$ connecting F' and F''. Let α be any essential curve in $F' - \eta(T)$ with both endpoints at $\lambda \cap F'$. As F' and F'' are T-parallel to the same component of $\partial_- C$, they are T-parallel to each other. This parallelism gives a rectangle D with two opposite sides running along λ, one side in α and the last side is the image of α in F'' under the parallelism. As D does not intersect T and therefore doesn't intersect any of the tubes in Λ, it is a T-compressing disk for F, a contradiction. Therefore all regions of T-parallelism between components of $\mathscr{F}$ and components of $\partial_- C$ are disjoint from all other components of $\mathscr{F}$.

Consider now a sphere component S of $\mathscr{F}$ bounding a ball B. If B contains any other component S' of $\mathscr{F}$, this component must be a sphere T-parallel to S. Note that as B contains a vertex, S and S' have at least three punctures each. We can now repeat the argument above to show that F is T-compressible. Therefore we conclude that each component F' of $\mathscr{F}$ bounds a compression body $C_{F'}$ which is either a ball intersecting T in a single vertex or is homeomorphic to $F' \times I$ and intersects T in vertical arcs. All of these compression bodies are disjoint and are also disjoint from all tubes Λ. This also implies that each tube in Λ is parallel to a different edge of Γ. The union of the compression bodies and the parallelisms between each tube in Λ and some edge of Γ gives the desired compression body C_F. □

Finally we recall the classification of Heegaard splittings of pairs (C,T) where $\partial_+ C$ is T-parallel to $\partial_- C$, [TT]. This result is key to proving our main theorem.

Theorem 3.5. [TT, Theorem 3.1] *Let M be a compression body and T be a properly embedded graph so that $\partial_+ M$ is T-parallel to $\partial_- M$. Let H be a Heegaard surface for (M,T). Assume that T contains at least one edge. Then one of the following occurs:*

(1) *H is stabilized;*
(2) *H is boundary stabilized;*
(3) *H is perturbed;*
(4) *T has a removable path disjoint from $\partial_+ M$;*
(5) *M is a 3–ball, T is a tree with a single interior vertex (possibly of valence 2), and $H - \mathring{\eta}(T)$ is parallel to $\partial M - \mathring{\eta}(T)$ in $M - \mathring{\eta}(T)$;*
(6) *$M = \partial_- M \times I$, H is isotopic in $M - \mathring{\eta}(T)$ to $\partial_+ M - \mathring{\eta}(T)$.*

For our purposes we need to strengthen the second conclusion of the above theorem:

Theorem 3.6. *Let M be a compression body and T be a properly embedded graph so that $\partial_+ M$ is T-parallel to $\partial_- M$. Let H be a Heegaard surface for (M,T). Assume that T contains at least one edge. Then one of the following occurs:*

(1) *H is stabilized;*
(2) *H is boundary stabilized along $\partial_- M$;*
(3) *H is perturbed;*
(4) *T has a removable path disjoint from $\partial_+ M$;*
(5) *M is a 3–ball, T is a tree with a single interior vertex (possibly of valence 2), and $H - \mathring{\eta}(T)$ is parallel to $\partial M - \mathring{\eta}(T)$ in $M - \mathring{\eta}(T)$;*
(6) *$M = \partial_- M \times I$, H is isotopic in $M - \mathring{\eta}(T)$ to $\partial_+ M - \mathring{\eta}(T)$.*

Proof. Suppose H is boundary stabilized along $\partial_+ M$. Then H is obtained by amalgamating a minimal genus Heegaard surface for $\partial_+ M \times [-1,0]$ which does not separate $\partial_+ M \times \{-1\}$ and $\partial_+ M \times \{0\}$ and which intersects each edge in $T \cap (\partial_+ M \times [-1,0])$ exactly twice, together with a Heegaard surface $\tilde{H}$ for M. Without loss of generality we will assume that H is obtained from $\tilde{H}$ after a single boundary stabilization along $\partial_+ M$.

By Theorem 3.5, $\tilde{H}$ satisfies one of six possible conclusions. If $\tilde{H}$ is stabilized, perturbed or if T has a removable path disjoint from $\partial_+ M$ then the same is true for H as boundary stabilizations preserve all of these properties. If $\tilde{H}$ is boundary stabilized, the stabilization must be along $\partial_- M$, and so the same is true for H.

Suppose that M is a 3–ball, T is a tree with a single interior vertex, and $H - \mathring{\eta}(T)$ is parallel to $\partial M - \mathring{\eta}(T)$ in $M - \mathring{\eta}(T)$. Let A and B be the components of $M - \tilde{H}$ so that B is a ball and let κ be one of the edges of $T \cap M$. Then H can be recovered from $\tilde{H}$ by tubing H to the boundary of a collar of ∂M along a vertical tube τ in A. We can choose τ to be arbitrarily close to $\kappa \cap A$; in particular, we may assume that the disk of parallelism between τ and κ intersects some bridge disk that contains $\kappa \cap B$ in its boundary only in the point $\kappa \cap \tilde{H}$. We conclude that H is perturbed.

Suppose then that $M = \partial_- M \times I$ and H is isotopic in $M - \mathring{\eta}(T)$ to $\partial_+ M - \mathring{\eta}(T)$. Let A and B be the components of $M - \tilde{H}$ so that A contains $\partial_+ M$. The argument in this case is identical to the one above as long as there is at least one bridge disk in B. If $T \cap B$ is a product, then $T \cap M$ is a collection of vertical arcs and thus H can be obtained from the Heegaard surface $\tilde{H}$ by stabilizing along $\partial_- M$. □

4. HAKEN'S LEMMA

Suppose (M,T) is a 3-manifold containing a properly embedded graph. Let H be a Γ-bridge surface for (M,T,Γ) and suppose D is a T-compressing disk for some component G of ∂M. It is a classic result of Haken [H] that in the case $T = \emptyset$, there is a compressing disk D' for G so that D' intersects H in a unique essential curve. This result was extended to the case where T is a tangle and $\Gamma = \emptyset$ in [HS3] and to the case where T is a tangle and $\Gamma = T$ in [T1]. In this paper we will need a further extension of the result to include properly embedded graphs. It turns out that the proof of [T1, Theorem 6.2] carries over to this situation without any modifications so we will not include it here.

Theorem 4.1. *Suppose M is a compact orientable manifold, T is a properly embedded graph in M and Γ is a subgraph of T. Assume that $M - T$ is irreducible and that no sphere in M intersects T transversally exactly once. Suppose H is a Γ-Heegaard splitting for (M,T,Γ). If D is a T-compressing disk for some component of ∂M then there exists such a disk D' so that D' intersects $H - \mathring{\eta}(T)$ in a unique essential simple closed curve.*

5. MULTIPLE Γ-BRIDGE SPLITTINGS OF (M,T,Γ)

5.1. **Complexity.**

Definition 5.1. *Let X be a set with an order $\leq$. Let Y and Z be two finite multisets of elements of X. Write $Y = (y_1, y_2, \dots, y_n)$ and $Z = (z_1, z_2, \dots, z_m)$ so that for all i, $y_i \geq y_{i+1}$ and $z_i \geq z_{i+1}$. We say that $Y < Z$ if and only if one of the following occurs:*

- *There exists $j \leq \min(n, m)$ so that for all $i < j$, $y_i = z_i$ and $y_j < z_j$.*
- *$n < m$ and for all $i \leq n$, $y_i = z_i$.*

The set of all Γ-multiple bridge splittings for (M, T, Γ) can be ordered using the definition of complexity introduced in [T1].

Definition 5.2. *Let S be a closed connected surface embedded in M transverse to a properly embedded graph $T \subset M$. The complexity of S is the ordered pair $c(S) = (2 - \chi(S - \eta(T)), \text{genus}(S))$. If S is not connected, $c(S)$ is the multi-set of ordered pairs corresponding to each of the components of S.*

If $\mathscr{H}$ is a Γ-multiple bridge splitting for (M, T, Γ), let the complexity of $\mathscr{H}$, $c(\mathscr{H})$ be the multiset $\{c(S) | S \in \mathscr{H}^+\}$. If $\mathscr{H}$ and $\mathscr{H}'$ are two multiple Γ-Heegaard splittings for (M, T, Γ), their complexities will be compared as in Definition 5.1.

Lemma 5.3. [T1, Lemma 4.4] *Suppose S is meridional surface in (M, T) of non-positive euler characteristic. If S' is a component of the surface obtained from S by compressing along a c-disk, then $c(S) > c(S')$.*

The next lemma is immediate from the definition of complexity.

Lemma 5.4. *Suppose that $\mathscr{H}$ is a multiple Γ-Heegaard splitting of (M, T, Γ) and suppose that $\mathscr{J}$ is a multiple Γ-Heegaard splitting of (M, T, Γ) such that $\mathscr{J}^+$ is a proper subset of $\mathscr{H}^+$. Then the complexity of $\mathscr{J}$ is strictly less than the complexity of $\mathscr{H}$.*

6. Untelescoping and Consolidation

We will be interested in obtaining a multiple Γ-Heegaard splitting of (M, T, Γ) with the property that every thin surface is incompressible. The following lemma will be useful. Its proof is straightforward and similar to the proof of [T1, Corollary 6.3].

Lemma 6.1. *Suppose that $\mathscr{H}$ is a multiple Γ-Heegaard splitting of (M, T, Γ). If some component of $\mathscr{H}^-$ is compressible in M then one of the following occurs:*

- *Some component of $\mathscr{H}^-$ is parallel to a component of $\mathscr{H}^+$; or*
- *Some component of $\mathscr{H}^+$ induces a (T, Γ)-c-weakly reducible Γ-Heegaard splitting of the component of $M - \mathscr{H}^-$ containing it.*

The next two subsections present two operations; each corresponds to one of the two possible conclusions in Lemma 6.1.

6.1. **Untelescoping.** In [ST1] Scharlemann and Thompson discussed the operation of *untelescoping* a weakly reducible Heegaard surface for a manifold to obtain a multiple Heegaard surface (i.e., generalized Heegaard splitting). The concept was generalized to weakly reducible bridge surfaces for a manifold containing a properly embedded tangle in [HS3] and then to (τ,τ)-c-weakly reducible τ-bridge surfaces for a manifold containing a properly embedded tangle τ in [T1]. In the following definition we extend the construction to (T,Γ)-c-weakly reducible Γ-bridge surfaces for (M,T,Γ) where T is a properly embedded graph.

Let H be a (T,Γ)-c-weakly reducible Γ-bridge splitting of (M,T,Γ) and let $\mathscr{D}_1$ and $\mathscr{D}_2$ be collections of pairwise disjoint (T,Γ)-c-disks above and below H such that $\mathscr{D}_1 \cap \mathscr{D}_2 = \emptyset$. Then we can obtain a multiple Γ-bridge splitting for (M,T) with one thin surface obtained from H by (T,Γ)-c-compressing it along $\mathscr{D}_1 \cup \mathscr{D}_2$ and two thick surfaces, obtained by (T,Γ)-c-compressing H along $\mathscr{D}_1$ and $\mathscr{D}_2$ respectively. Although this operation is similar to the one described in [T1] we include the details here with the modifications required.

Let $(A, A\cap T)$ and $(B, B\cap T)$ be the two Γ-compression bodies into which H decomposes (M,T,Γ) and let $\mathscr{D}_A \subset A$ and $\mathscr{D}_B \subset B$ be collections of pairwise disjoint (T,Γ)-c-disks such that $\mathscr{D}_A \cap \mathscr{D}_B = \emptyset$. Let $A' = A - \mathring{\eta}(\mathscr{D}_A)$ and $B' = B - \mathring{\eta}(\mathscr{D}_B)$. Then by Lemma 3.1 A' and B' are each the disjoint union of Γ-compression bodies containing trivial graphs $A'\cap T$ and $B'\cap T$ respectively.

Take small collars $\eta(\partial_+A')$ of ∂_+A' and $\eta(\partial_+B')$ of ∂_+B'. Let $C^1 = cl(A' - \eta(\partial_+A'))$, $C^2 = \eta(\partial_+A') \cup \eta(\mathscr{D}_B)$, $C^3 = \eta(\partial_+B') \cup \eta(\mathscr{D}_A)$ and $C^4 = cl(B' - \eta(\partial_+B'))$. Note that C_1 and C_4 are Γ-compression bodies containing trivial graphs because they are homeomorphic to A' and B' respectively. C_2 and C_3 are obtained by taking surface $\times I$ containing vertical arcs and attaching 2-handles, some of which may contain segments of Γ as their cores. Therefore C_2 and C_3 are also Γ-compression bodies containing trivial graphs. We conclude that we have obtained a multiple Γ-bridge splitting $\mathscr{H}$ of (M,T,Γ) with positive surfaces ∂_+C_1 and ∂_+C_2 that can be obtained from H by (T,Γ)-c-compressing along $\mathscr{D}_A$ and $\mathscr{D}_B$ respectively and a negative surface $\partial_-C_2 = \partial_-C_3$ obtained from H by (T,Γ)-c-compressing along both sets of c-disks. We say that $\mathscr{H}$ is obtained by untelescoping H using (T,Γ)-c-disks. The next remark follows directly from Lemma 5.3.

Remark 6.2. *Suppose $\mathscr{H}'$ is a multiple Γ-bridge splitting of (M,T,Γ) obtained from another multiple Γ-bridge splitting $\mathscr{H}$ of (M,T,Γ) via untelescoping. Then $c(\mathscr{H}') < c(\mathscr{H})$.*

Suppose that H is a multiple Γ-Heegaard splitting for (M,T,Γ) which can be untelescoped to become a multiple Γ-Heegaard splitting $\mathscr{H}$ for

(M,T,Γ). It is natural to ask: If $\mathscr{H}$ contains a generalized stabilization must H contain a generalized stabilization? If $\mathscr{H}$ contains a perturbed thick surface must H also be perturbed? If T has a path which is removable with respect to $\mathscr{H}$, is that path removable with respect to H? The answer is positive in all cases.

Lemma 6.3. *Let $\mathscr{H}$ be a multiple Γ-Heegaard splitting for the triple (M,T,Γ) obtained by untelescoping the (multiple) Γ-Heegaard splitting H for (M,T,Γ).*

(1) *Suppose the Γ-Heegaard splitting induced by $\mathscr{H}^+$ on some component of $(M,T,\Gamma)-\mathscr{H}^-$ contains a generalized stabilization. Furthermore suppose that if the generalized stabilization is a (meridional) boundary stabilization, then it is along a component of ∂M. Then H contains a generalized stabilization of the same type and if the generalized stabilization is a (meridional) boundary stabilization then it must also be along a component of ∂M.*
(2) *If the Γ-Heegaard splitting induced by $\mathscr{H}^+$ on some component of $(M,T,\Gamma)-\mathscr{H}^-$ is perturbed, then so is H.*
(3) *If the Γ-Heegaard splitting induced by $\mathscr{H}^+$ on some component of $(M,T,\Gamma)-\mathscr{H}^-$ contains a removable path so that the path is either a cycle or both of its endpoints are contained in ∂M, then so does H.*

Proof. Without the loss of much generality, we may assume that $\mathscr{H}$ is obtained from H by a single untelescoping operation. Suppose that H decomposes (M,T) into compression bodies A and B. Let N_i for $i=1,2$ be the closure of the components of $M-\mathscr{H}^-$ on either side of $\mathscr{H}^-$. Let H_i be the Γ-Heegaard surface of N_i induced by $\mathscr{H}^+$. Assume that H_1 contains a generalized stabilization, or is perturbed, or that $T\cap N_1$ contains a removable path satisfying the hypotheses of (3).

Consider a collar $\mathscr{H}^-\times[-1,1]$ of $\mathscr{H}^-$ where $\mathscr{H}^-=\mathscr{H}^-\times\{0\}$. Recall that H_1 can be obtained from $\mathscr{H}^-\times[0,1]$ by attaching handles (possibly with cores running along the knot) to $\mathscr{H}^-\times\{1\}$. Similarly, H_2 can be obtained from $\mathscr{H}^-\times[-1,0]$ by attaching handles to $\mathscr{H}^-\times\{-1\}$. The Heegaard surface H can be obtained by extending the handles of H_1 through $\mathscr{H}^-\times[-1,1]$ and attaching them to H_2.

Case 1A: H_1 is stabilized or meridionally stabilized. Let D and E be the (T,Γ)-c-disks with boundary on H_1 defining the (meridional) stabilization. We may assume that the handles attached to $\mathscr{H}^-\times\{1\}$ include one that has ∂D as a core. The intersection of ∂E with $\mathscr{H}^-\times\{1\}$ then consists of a single arc. The possibly once punctured disk $E'=E\cup(\partial E\times[-1,1])$ is then a c-disk with boundary on H. The c-disks E' and D intersect exactly once and define a (meridional) stabilization of H. See Figure 1.

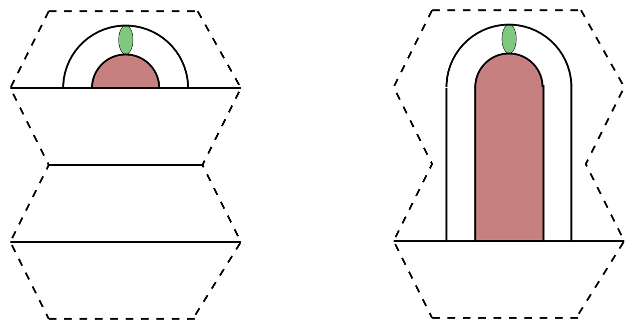

FIGURE 1. If H_1 is stabilized, so is H.

Case 1B: H_1 is boundary stabilized or meridionally boundary stabilized. Let C_1^+ and C_2^+ be the two Γ-compression bodies into which H_1 divides N_1. Without loss of generality in this case C_1^+ contains some component $G \subset \partial N$ and a c-disk D so that removing $\eta(D)$ from C_1^+ decomposes it into a compression body $C_1'^+$ and a component R homeomorphic to $G \times I$ and adding R to C_2^+ results in a compression body $C_2'^+$. Let $\tilde{H}_1 = \partial_+ C_1'^+ = \partial_+ C_2'^+$. Amalgamating the multiple bridge splitting with thick surfaces $\tilde{H}_1$ and H_1 gives a bridge splitting for N with bridge surface $\tilde{H}$ which can be obtained from H by c-compressing along the disk D. Therefore H is boundary stabilized or meridionally boundary stabilized.

Case 2: H_1 is perturbed. Let D and E be the two bridge disks for H_1 that intersect in one or two points and these points lie in T. Both disks are completely disjoint from $\mathscr{H}^- \times \{1\}$ so in particular extending the 1-handles from H_1 across $\mathscr{H}^- \times [-1,1]$ has no effect on these disks. Therefore H is also perturbed.

Case 3: Suppose ζ is a removable path in N_1 and suppose first that ζ is a cycle. Let C_1 and C_2 be the two Γ-compression bodies $\mathrm{cl}(N_1 - H_1)$. As ζ is a removable cycle there exists a cancelling pair of disks $\{D_1, D_2\}$ for ζ with $D_j \subset C_j$. These disks are both completely disjoint from $\mathscr{H}^- \times \{1\}$ so in particular extending the 1-handles from H_1 across $\mathscr{H}^- \times [-1,1]$ has no effect on these disks. As ζ is a removable cycle there exists a compressing disk E in C_1, say, so that $|E \cap T| = 1$ and $|\partial E \cap \partial D_2| = 1$ and E is otherwise disjoint from a complete collection of bridge disks for $(T \cap N_1) - H_1$ containing $D_1 \cup D_2$. We may assume that the handles attached to $\mathscr{H}^- \times \{1\}$ include one that has ∂E as a core. Therefore E satisfies all the desired properties as a compressing disk for H.

Suppose then that ζ is a path. Again we may assume that the handles attached to $\mathscr{H}^- \times \{1\}$ include one that has ∂E as a core. The bridge disk for the component of $\zeta - H_1$ is also a bridge disk for $\zeta - H$ satisfying all of the required properties. $\square$

6.2. **Consolidation.** Suppose that $\mathscr{H}$ is a multiple Γ-Heegaard surface for (M,T,Γ) and that there is a component $F \subset \mathscr{H}^-$ which is parallel to a component H of $\mathscr{H}^+$. That is, F and H cobound a submanifold C homeomorphic to $F \times I$ such that $C \cap T$ consists of vertical edges. Recall that $\partial_- C = F$ and $\partial_+ C = H$. Let $\mathscr{H}' = \mathscr{H} - (F \cup H)$. If the compression body of $M - \mathscr{H}$ adjacent to F but not to H does not contain vertical pods with handles adjacent to F, we say that $\mathscr{H}'$ is obtained from $\mathscr{H}$ by *consolidation*.

Lemma 6.4. *Suppose that $\mathscr{H}'$ is obtained from $\mathscr{H}$ by consolidation. Then $\mathscr{H}'$ is a multiple Γ-Heegaard splitting of (M,T,Γ). Furthermore, the complexity of $\mathscr{H}'$ is strictly less than the complexity of $\mathscr{H}$.*

Proof. Let C_+ be the compression body adjacent to $\partial_+ C$ and let C_- be the compression body adjacent to $\partial_- C$. We can view C_- as obtained by adding 1-handles to $\partial_- C$ where some of these 1-handles may have segments of Γ as their cores. These 1-handles can be extended through the product structure of C to be considered as added to $\partial_+ C = \partial_+ C_+$ and then they can be further extended to be added to $\partial_- C_+$. Thus, the union of C, C_-, and C_+ is a compression body. Since $T \cap C$ is a collection of vertical edges and since $T \cap C_+$ contains no vertical pods with handles adjacent to F, $T \cap (C_- \cup C \cup C_+)$ is trivially embedded in $C_- \cup C \cup C_+$. By Lemma 5.4, the complexity of $\mathscr{H}'$ is strictly less than the complexity of $\mathscr{H}$. □

6.3. **Combining untelescoping and consolidation.** We will usually use consolidation in conjunction with untelescoping, so the next lemma is important.

Lemma 6.5. *Suppose that $\mathscr{H}$ is a multiple Γ-Heegaard surface obtained by untelescoping and consolidating a Γ-Heegaard surface H for (M,T,Γ). If e is an edge in $M - \mathscr{H}$ which is a pod handle then e is a pod handle in $M - H$. In particular, e is adjacent to ∂M.*

Proof. First note that consolidation never introduces pod-handles that are adjacent to a thin surface.

Let A and B be the compression bodies which are the closures of the components of $M - H$. Let D be a (T,Γ)-c-disk in A. Let A' be obtained by reducing A using D and let H' be a copy of $\partial_+ A'$ pushed slightly into A'. The surface H' is a Γ-Heegaard surface for A'. Let A'_1 be the compression body of $A' - H'$ adjacent to $\partial_+ A'$. Each component of $T \cap A'$ is a vertical edge, since, in creating H', we did not push $\partial_+ A'$ past any vertices of $T \cap A'$.

Suppose then that E is a (T,Γ)-c-disk in B. Let A'' be obtained from A' by attaching a regular neighborhood of E to $\partial_+ A'$. The surface H' is still a Γ-Heegaard surface for A''. If E was a T-compressing disk then

$T \cap A'' = T \cap A'$. If E was a (T,Γ)-cut disk then $T \cap A''$ contains one more edge than $T \cap A'$. The new edge is disjoint from H and has both endpoints on $\partial A''$. It is, therefore, not a pod handle in $A'' - H'$. The lemma follows immediately from these observations. □

Suppose that $\mathscr{J}$ is a multiple Γ-Heegaard splitting of (M,T,Γ) obtained by untelescoping a Γ-Heegaard surface H for (M,T,Γ). If a component of $\mathscr{J}^-$ is parallel to a component of $\mathscr{J}^+$ then we may consolidate $\mathscr{J}$ to obtain a multiple Γ-Heegaard splitting $\mathscr{K}$ of (M,T,Γ) such that no component of $\mathscr{K}^-$ is parallel to a component of $\mathscr{K}^+$. The proof of the next lemma is similar to that of Lemma 6.3 and so we omit it.

Lemma 6.6. *Use the above notation.*

- *Suppose the Γ-Heegaard splitting induced by $\mathscr{K}^+$ on some component of $(M,T,\Gamma) - \mathscr{K}^-$ contains a generalized stabilization. Furthermore, suppose that if the generalized stabilization is a (meridional) boundary stabilization, then it is along a component of ∂M. Then H contains a generalized stabilization of the same type and if the generalized stabilization is a (meridional) boundary stabilization then it must also be along a component of ∂M.*
- *If the Γ-Heegaard splitting induced by $\mathscr{K}^+$ on some component of $(M,T,\Gamma) - \mathscr{K}^-$ is perturbed then so is H.*
- *If the Γ-Heegaard splitting induced by $\mathscr{K}^+$ on some component of $(M,T,\Gamma) - \mathscr{H}^-$ contains a removable path which is either a cycle or which has both endpoints in ∂M, then so does H.*

7. Untelescoping and Essential Surfaces

Theorem 7.1. *Let M be a compact, orientable 3-manifold containing a properly embedded graph T and let Γ be a subgraph of T. Furthermore assume that $M - T$ is irreducible and that no sphere in M intersects T exactly once. Suppose H is a Γ-bridge surface for (M,T) that is (T,Γ)-c-weakly reducible and T-irreducible. Then there is a multiple Γ-bridge splitting for (M,T,Γ) such that*

- *$\mathscr{H}^-$ is incompressible in the complement of T;*
- *no component of $\mathscr{H}^-$ is parallel to a component of $\mathscr{H}^+$;*
- *each component of $\mathscr{H}^+$ is (T,Γ)-c-strongly irreducible in $M - \mathscr{H}^-$; and*
- *$\mathscr{H}$ is obtained from H by untelescoping and consolidation (possibly many times).*

Proof. Untelescope H as described in Section 6.1 along a collection of (T,Γ)-c-disks. Let $\mathscr{H}_1$ be the resulting Γ-bridge surface with a single (possibly disconnected) thin surface. We will define $\mathscr{H}_i$ for $i \geq 2$ inductively as follows: If components of $\mathscr{H}_{i-1}^-$ are parallel to components of $\mathscr{H}_{i-1}^+$, by Lemma 6.5, we may consolidate $\mathscr{H}_{i-1}$ to obtain a multiple Γ-Heegaard splitting $\mathscr{H}_i$ of (M,T,Γ). If no component of $\mathscr{H}_{i-1}^-$ is parallel to a component of $\mathscr{H}_{i-1}^+$ but a component of $\mathscr{H}_{i-1}^+$ is (T,Γ)-c-weakly reducible in the component of $M-\mathscr{H}_{i-1}^-$ containing it, then we may untelescope $\mathscr{H}_{i-1}$ to create $\mathscr{H}_i$.

Since both untelescoping and consolidation strictly reduce complexity, eventually this process terminates with a Γ-multiple Heegaard splitting $\mathscr{H}$ of (M,T,Γ) such that:

- No component of $\mathscr{H}^-$ is parallel to any component of $\mathscr{H}^+$; and
- Each component of $\mathscr{H}^+$ is (T,Γ)-c-strongly irreducible in the component of $M-\mathscr{H}^-$ containing it.

By Lemma 6.1, $\mathscr{H}^-$ is incompressible. □

We can now prove our main theorem.

Theorem 7.2. *Let M be a compact, orientable 3-manifold containing a properly embedded graph T and let Γ be a subgraph of T. Furthermore assume that $M-T$ is irreducible and no sphere in M intersects T exactly once. Let H be a (T,Γ)-c-weakly reducible Heegaard surface for (M,T). Then one of the following holds:*

- *there is a multiple Γ-Heegaard splitting $\mathscr{H}$ for (M,T,Γ) so that each thin surface is T–essential and each thick surface is (T,Γ)-c-strongly irreducible in the component of $M-\mathscr{H}^-$ containing it,*
- *H contains a generalized stabilization,*
- *H is perturbed, or*
- *T has a removable path.*

Proof. Let $\mathscr{H}$ be the multiple Γ-Heegaard splitting of (M,T,Γ) provided by Theorem 7.1.

Suppose that a component F of $\mathscr{H}^-$ is T-parallel to ∂M. Let C_F be the compression body so that $F=\partial_+C_F$ is parallel to the boundary of a regular neighborhood of some components of ∂M together with some subset of T. By Lemma 3.4 we may assume that F is innermost, i.e., C_F does not contain any other thin surfaces. Let H_{C_F} be the Γ-Heegaard splitting for C_F given by the unique thick surface of $\mathscr{H}$ contained in C_F.

Case 1: $T\cap C_F=\varnothing$.

Recall that F is not parallel to H_{C_F}. If $\partial_-C_F=\varnothing$, then by [W], H is stabilized. If $\partial_-C_F\neq\varnothing$, by [ST3] H_{C_F} is stabilized or boundary stabilized along ∂_-C_F. See also [MS]. By Lemma 6.6, H contains a generalized stabilization.

Case 2: H_{C_F} is a Heegaard splitting.

Since no thin surface is parallel to a thick surface, by Theorem 3.6 one of the following occurs:

- H_{C_F} has a generalized stabilization and if this is a boundary stabilization, it is along a component of ∂M;
- H_{C_F} is perturbed; or
- $T \cap C_F$ has a removable path disjoint from F.

By Lemma 6.6, H is either perturbed or it contains a generalized stabilization or a removable path as desired.

Case 3: H_{C_F} is a Γ-Heegaard splitting but not a Heegaard splitting.

Let A and B be the two compression bodies into which H_{C_F} divides C_F. Since H_{C_F} is a Γ-Heegaard splitting but not a Heegaard splitting, there exists an edge $e \subset \Gamma$ in either A or B which is disjoint from H_{C_F} and which has both endpoints on ∂C_F.

Case 3a: $\partial e \subset F$.

Since F is T-parallel to $\partial M \cup T$ and since $e \subset T$ is an edge with both endpoints on F, $T = e$ and $F = S^2$. Then by [HS2, Lemma 2.1] and [HS1, Theorem 1.1], either H_{C_F} is stabilized, meridionally stabilized, or perturbed. See Case 2 of the proof of [T1, Lemma 5.2] for details. By Lemma 6.6, H is stabilized, meridionally stabilized, or perturbed.

Case 3b: $\partial e \subset \partial M$

In this case e is disjoint not only from H_{C_F} but also from H. Then e is an edge of Γ with both endpoints on ∂M which is disjoint from H, contrary to the hypothesis that H is a Heegaard surface.

Case 3c: One endpoint of e is on F and one endpoint of e is on ∂M.

We may assume that the hypotheses of cases (3a) and (3b) do not apply. Let $e_1, \dots, e_n$ be the union of edges of $T \cap C_F$ with one endpoint on F, one endpoint on ∂M, and which are disjoint from H_{C_F}. Perform a slight isotopy of each of them to convert T into a graph T' and each edge e_i into e_i' so that each e_i' is a removable edge of T' as in Lemma 2.2. Let H_{C_F}' be the new Γ-Heegaard surface and notice that H_{C_F}' is, in fact, a Heegaard surface for C_F. Since F is not parallel to H_{C_F}, by Theorem 3.6 one of the following occurs:

- H_{C_F}' has a generalized stabilization and if this is a boundary stabilization, it is along a component of ∂M;
- H_{C_F}' is perturbed; or
- $T \cap C_F$ has a removable path with both endpoints in ∂M.

Notice that if the last option occurs the removable path is not equal to any of the e_i' since each of those edges has one endpoint on F. By Lemma 2.2 one of the following occurs:

- H_{C_F} has a generalized stabilization and if this is a boundary stabilization, it is along a component of ∂M;
- H_{C_F} is perturbed; or
- $T \cap C_F$ has a removable path disjoint from F.

By Lemma 6.6, either H contains a generalized stabilization, or H is perturbed, or T contains a removable path. □

8. Intersections between a surface and a bridge surface

Let T be a finite trivalent graph containing at least one edge embedded in a closed orientable 3–manifold M. Let H be a Heegaard surface for (M,T). We will consider the intersections between H and a surface F in the exterior of the graph. Throughout we let $\Gamma = T$ and write "c-weakly reducible" for "T–c-weakly reducible", etc.

The result in this section will be used in the proof of Theorem 9.1; however, it is also of interest in its own right. The main idea of the proof can be traced back to Gabai's proof of the Poenaru conjecture [G]. Similar ideas were also key to the results in [GST].

Theorem 8.1. *Suppose that F is a surface properly embedded in the exterior of T. Assume that ∂F is essential in $\partial\eta(T)$ and that at least one component of ∂F is not a meridian of T . Then one of the following is true:*

(1) *$H - T$ is c-weakly reducible in $M - T$.*
(2) *$M = S^3$, $H = S^2$, and T is the unknot in 1–bridge position with respect to H.*
(3) *H is bimeridionally stabilized.*
(4) *H is perturbed.*
(5) *T has a perturbed level cycle and the associated cancelling pair of disks lie in F.*
(6) *H can be isotoped so that $F \cap H$ is a non-empty collection of arcs and circles and so that no arc or circle of intersection is inessential in F.*

8.1. Normal Form.

As in [GST, ST4, G] the main tool will be a sweepout of M by H and an examination of the upper and lower disks for T in F. We briefly recall the central concepts.

Let $C_\uparrow$ and $C_\downarrow$ be the handlebodies on either side of H in M. Removing spines for $C_\uparrow$ and $C_\downarrow$ from M creates a manifold homeomorphic to $H \times (0,1)$. Let $h\colon H \times (0,1)$ be projection onto $(0,1)$ and extend h to be a height function $h\colon M \to [0,1]$. The inverse image of $t \in (0,1)$ is a surface H_t isotopic to H. We choose the labelling of $C_\uparrow$ and $C_\downarrow$ so that for a specified H_t, $h^{-1}[t,1)$ is $C_\uparrow$ and $h^{-1}(0,t]$ is $C_\downarrow$. That is, $C_\uparrow$ lies above $C_\downarrow$. Isotope h so that T is in *normal form* with respect to h [S, Def. 5.1]. This means that

(a) if e is an edge of T, the critical points of $h|_e$ are nondegenerate and each lies in the interior of an edge.
(b) the critical points of $h|_{\text{edges}}$ and the vertices of T all occur at different heights, and
(c) at each trivalent vertex v of T either two ends of incident edges lie above v (so that v is a y-vertex) or two ends of incident edges lie below v (so that v is a λ-vertex.)

T can be perturbed by a small isotopy to be normal. We will always assume that T is in normal form with respect to h. The maxima of T consist of all local maxima of $h|_{\text{edges}}$ and all λ-vertices. The minima of T consist of all local minima of $h|_{\text{edges}}$ and all y-vertices. The *critical points* of T are all maxima and minima; the heights of the critical points are the *critical values*. Notice that since T is in bridge position with respect to H, all maxima are above all minima and we can interchange by an isotopy of T rel H the heights of two maxima or the heights of two minima.

If $F \subset M - \mathring{\eta}(T)$ is a properly embedded surface, F is in *normal form* with respect to h [S, Def. 5.6] if

(1) Each critical point of h on F is non-degenerate;
(2) Each component of ∂F on $\partial\eta(T)$ is either a horizontal meridional circle or contains only non-degenerate critical points, and occurs near an associated critical point of T in $\partial\eta(T)$; the number of critical points has been minimized up to isotopy;
(3) No critical point of h on $\mathring{F}$ occurs near a critical height of h on T;
(4) No two critical points of h on $\mathring{F}$ or ∂F occur at the same height;
(5) The minima (resp. maxima) of $h|_{\partial F}$ at the minima (resp. maxima) of T are half-center singularities; and
(6) The maxima of $h|_{\partial F}$ at y and λ-vertices are half-saddle singularities of h on F.

The surface F can always be properly isotoped to be in normal form.

8.2. **Bridge and loop boundary compressing disks and their associated cut and compressing disks.** Suppose that $t \in (0,1)$ and that D is a compressing disk or ∂ compressing disk for $H_t - \mathring{\eta}(T)$ in $M - T$. If D lies above H_t, we call D an *upper disk* and if D lies below H_t, we call D a *lower disk*. For certain values of t, an upper or lower disk has a specified form. Let $t_{\min}$ be the height of the highest minimum of $h|_T$ and let $t_{\max}$ be the height of the lowest maximum of $h|_T$.

Suppose that D is an upper or lower disk for H_t with $t \in (t_{\min}, t_{\max})$. If D is a ∂-compressing disk for $H - T$ such that the arc $\partial D \cap H$ has endpoints in distinct components of $\partial(H - \mathring{\eta}(T))$ then D is called a *bridge boundary* *compressing disk*. See Figure 2. If the endpoints of $\partial(D \cap H)$ lie in the same component of $\partial(H - \mathring{\eta}(T))$, then D is a *loop boundary compressing disk*.

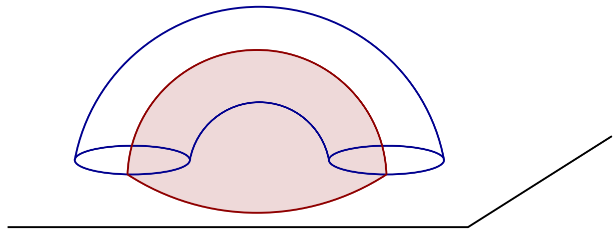

FIGURE 2. An example of a bridge boundary compression

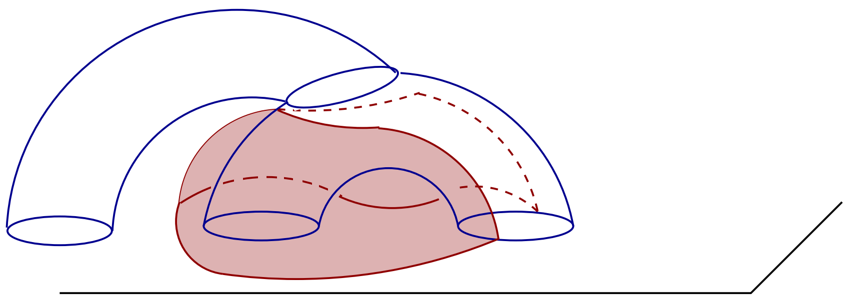

FIGURE 3. An example of a loop boundary compression

If D is a ∂-compressing disk for $H - \mathring{\eta}(T)$, ∂D can be extended so that the interior of D lies in $M - T$ and ∂D consists of an arc in H and an arc in T. We let $\overline{D}$ denote the extended disk. If D is a bridge boundary compressing disk, then the frontier of a regular neighborhood of $\overline{D}$ in $C_\uparrow$ or $C_\downarrow$ is a disk D' which has boundary in $H - T$ and intersects T in 0 or 1 points. If D' is disjoint from T, then either D' is a compressing disk for $H - T$ or $H = S^2$, $|T \cap H| = 2$, and T is the unknot in $M = S^3$. If $|D' \cap T| = 1$, then either D' is a cut disk for $H - T$, or $H = S^2$, $|T \cap H| = 3$, and T is a connected graph with two vertices and three edges. If D' is a compressing disk or cut disk for $H - T$, then we say that D' is *associated* to D and $\overline{D}$.

If D is a loop boundary compressing disk, $\partial\overline{D}$ consists of a loop in H. The frontier of a regular neighborhood of $\overline{D}$ consists of two disks, D'_1 and D'_2. Each of D'_1 and D'_2 intersects T exactly once.

Lemma 8.2. *If neither D'_1 nor D'_2 is a cut disk, then $H = S^2$ and $|T \cap H| = 3$.*

Proof. Without loss of generality, suppose that $D \subset C_\uparrow$. Since neither D'_1 nor D'_2 is a cut disk each of their boundaries must be parallel in $H - \mathring{\eta}(T)$ to a component of $\partial(H - \mathring{\eta}(T))$. In fact, if D'_j intersects an edge e of $T \cap C_\uparrow$, then $\partial D'_j$ is isotopic to $\partial\eta(e)\cap H$. By construction, D'_1 and D'_2 intersect distinct edges of $T\cap C_\uparrow$. Thus, $H-(\partial D'_1\cup\partial D'_2)$ consists of three components: two disks, each once punctured by T, and an annulus A punctured once by T. (The puncture $T\cap A$ is the point $T\cap\partial\overline{D}\cap H$.) Thus, $H=S^2$, and $|T\cap H|=3$. □

Lemma 8.3. *Suppose that D'_1 is not a cut disk for $H-T$ and that $D\subset C_\uparrow$. Then there exists a bridge disk E for $T\cap C_\uparrow$ such that the edge of $T\cap C_\uparrow$ lying in $\overline{D}$ is contained in ∂E and any properly embedded arc in $H-T$ which has been isotoped to intersect $\partial E\cup\partial\overline{D}$ minimally intersects ∂E if and only if it intersects $\partial\overline{D}$.*

Proof. Since D'_1 is not a cut disk and since T is in bridge position with respect to H, there exists a 3–ball B embedded in $C_\uparrow$ such that $\partial B=D'_1\cup E'$ where E' is a disk in H punctured once by T. Let e be the edge of $T\cap C_\uparrow$ which lies in $\overline{D}$. The ball B can be extended to a ball B' such that $\overline{D}\subset\partial B'$, $\partial B'\cap H$ is a disk punctured once by T, and the interior of B' contains an edge e' of $T\cap C_\uparrow$. Since T is in bridge position with respect to H, there exists a bridge disk E for $T\cap C_\uparrow$ such that $\partial E\cap T=e\cup e'$ and $E\subset B'$. Since $B'\cap(H-T)$ is a once-punctured disk and since $\partial E\cap H$ joins $\partial(B'\cap H)$ to $T\cap B'\cap H$, any arc in $B'\cap(H-T)$ which intersects both ∂E and $\partial\overline{D}$ minimally intersects one if and only if it intersects the other. □

Proof of Theorem 8.1. Assume that $H-T$ is c-strongly irreducible in $M-T$ and that (2) does not occur.

Claim 1: If $H=S^2$ and if $|T\cap H|=3$, then T is perturbed.

Proof: Let v_1, v_2, and v_3 be the points of $T\cap H$. Let $D_\uparrow$ be a bridge disk for $T\cap C_\uparrow$ so that $D_\uparrow\cap H$ is an arc which joins v_1 and v_2. Let $D_\downarrow$ be a bridge disk for $T\cap C_\downarrow$ so that $D_\downarrow\cap H$ is an arc which joins v_2 and v_3. Since $H-T$ is a thrice-punctured sphere, the disks $D_\uparrow$ and $D_\downarrow$ can be isotoped in $H-T$ so that the arcs $D_\uparrow\cap H$ and $D_\downarrow\cap H$ have disjoint interiors in $H-T$. Then $\{D_\uparrow,D_\downarrow\}$ is a perturbing pair for H. □(Claim 1)

Since the conclusion that T is perturbed is one of the possible conclusions of Theorem 8.1, we assume from now on that if $H=S^2$, then $|T\cap H|\neq 3$.

Claim 2: Suppose that $t\in(t_{\min},t_{\max})$. Then after an isotopy of H to eliminate arcs and circles of intersection of $H\cap F$ which are inessential in both $H-\mathring{\eta}(T)$ and F, there is not simultaneously an upper disk and a lower disk for $H_t-\mathring{\eta}(T)$ such that both disks lie in F.

Proof: Suppose that there is an upper disk $D_\uparrow$ and lower disk $D_\downarrow$ for $H_t-\mathring{\eta}(T)$, such that both disks lie in F. If t is a critical value of $h|_F$, the interiors of $\partial D_\uparrow\cap H$ and $\partial D_\downarrow\cap H$ may not be disjoint. In this case, however, a small isotopy of one of them will make the interior of $\partial D_\uparrow\cap H$ disjoint from the interior of $\partial D_\downarrow \cap H$. We will always assume that this isotopy, if needed, has been performed. If $D_\uparrow$ or $D_\downarrow$ is a ∂-compressing disk, extend it to a disk $\overline{D}_\uparrow$ or $\overline{D}_\downarrow$ as before.

As the situation is symmetric with respect to $\uparrow$ and $\downarrow$ and there are three possibilities for each kind of disk: compressing, bridge ∂-compressing and loop ∂-compressing. There are six cases to consider; we will handle two of them: one disk a compressing disk and the other either kind of ∂-compressing disk, at the same time.

Case 1: Both $D_\uparrow$ and $D_\downarrow$ are compressing disks.

We may assume that $D_\uparrow$ and $D_\downarrow$ are disjoint. (If they were not a small isotopy, as above, would make them so.) Thus $H - T$ is c-weakly reducible, a contradiction.

Case 2: One disk, say $D_\uparrow$, is a compressing disk and $D_\downarrow$ is a ∂-compressing disk.

Since $D_\uparrow$ is a compressing disk, if $H = S^2$, then $|T \cap H| \geq 4$. Thus, if $D_\downarrow$ is a bridge boundary compressing disk, there is a c-disk $D'_\downarrow$ for $H - T$ associated to $\overline{D}_\downarrow$ and if $D_\downarrow$ is a loop boundary compressing disk, there is an associated cut-disk $D'_\downarrow$. In either case the boundaries of $D_\uparrow$ and $D_\downarrow$ are disjoint, so $D_\uparrow \cap D'_\downarrow = \emptyset$ and so $H - T$ is c-weakly reducible, a contradiction.

Case 3: Both $D_\uparrow$ and $D_\downarrow$ are bridge boundary compressing disks.

The arcs $\overline{D}_\uparrow \cap H$ and $\overline{D}_\downarrow \cap H$ have disjoint interiors. If they also have disjoint endpoints, then $|T \cap H| \geq 4$ and there are disjoint c-disks $D'_\uparrow$ and $D'_\downarrow$ associated to $D_\uparrow$ and $D_\downarrow$, respectively. In this case, $H - T$ is c-weakly reducible contradicting our assumption. If $\overline{D}_\uparrow \cap H$ and $\overline{D}_\downarrow \cap H$ share exactly one endpoint then H is perturbed, so assume that they share both endpoints. In this case, the cycle in T which is the closure of $(\partial \overline{D}_\uparrow \cap \partial \overline{D}_\downarrow) - H$, is a perturbed level cycle. Thus, T has a perturbed level cycle and the associated cancelling pair of disks lie in F.

Case 4: One of the disks, say $D_\uparrow$, is a bridge boundary compressing disk and $D_\downarrow$ is a loop boundary compressing disk.

Suppose, first, that $\partial \overline{D}_\uparrow$ and $\partial \overline{D}_\downarrow$ are disjoint. Let $D'_\uparrow$ be the c-disk associated to $\overline{D}_\uparrow$. Let $D^1_\downarrow$ and $D^2_\downarrow$ be the once-punctured disks associated to $\overline{D}_\downarrow$. If $H = S^2$, then $|T \cap H| \neq 3$, so we may assume that $D^1_\downarrow$, say, is a cut disk for $H - T$. Since $\overline{D}_\uparrow$ and $\overline{D}_\downarrow$ are disjoint, $D'_\uparrow$ and $D^1_\downarrow$ are disjoint and so $H - T$ is c-weakly reducible, a contradiction.

Thus, we may assume that $\partial \overline{D}_\uparrow \cap H$ and $\partial \overline{D}_\downarrow \cap H$ intersect. The intersection must be one of the endpoints of the arc $\partial \overline{D}_\uparrow \cap H$; let v be this point of $T \cap H$. One of $D^1_\downarrow$ or $D^2_\downarrow$ is disjoint from $\overline{D}_\uparrow$. Without loss of generality, suppose it to be $D^1_\downarrow$. If $D^1_\downarrow$ is a cut disk, then the disks $D'_\uparrow$ and $D^1_\downarrow$ show that

$H-T$ is c-weakly reducible, a contradiction. Thus, $D^1_\downarrow$ is not a cut disk. Let $E_\downarrow$ be the bridge disk provided by Lemma 8.3. Since $D^1_\downarrow$ is disjoint from $\overline{D}_\uparrow$, the interior of the arc $\partial E_\downarrow \cap H$ is disjoint from the interior of the arc $\overline{D}_\uparrow$. Notice that, by the construction of E from $D^1_\downarrow$, the arcs $\partial E_\downarrow \cap H$ and $\partial \overline{D}_\uparrow \cap H$ share exactly one endpoint. Thus, T is perturbed.

Case 5: Both $D_\uparrow$ and $D_\downarrow$ are loop boundary compressing disks.

Let D^1_j, D^2_j be the once-punctured disks associated to D_j for $j \in \{\uparrow,\downarrow\}$. Suppose, first, that $\overline{D}_\uparrow$ and $\overline{D}_\downarrow$ are disjoint. If $H = S^2$, then $|H\cap T| \neq 3$. Thus, one of D^1_j and D^2_j is a cut disk for $j \in \{\uparrow,\downarrow\}$. Without loss of generality, suppose that both $D^1_\uparrow$ and $D^1_\downarrow$ are cut disks. Since $\overline{D}_\uparrow$ and $\overline{D}_\downarrow$ are disjoint, the cut disks $D^1_\uparrow$ and $D^1_\downarrow$ are disjoint. Thus, $H-T$ is c-weakly reducible, a contradiction.

We may, therefore, assume that $\overline{D}_\uparrow$ and $\overline{D}_\downarrow$ intersect at a single point $v \in T\cap H$.

Case 5a: The endpoints of $\partial D_\uparrow$ on $\partial \eta(v) \subset H$ do not alternate with the endpoints of $\partial D_\downarrow$.

Choose the labelling of $D^1_\uparrow$, $D^2_\uparrow$ and $D^1_\downarrow$, $D^2_\downarrow$ carefully so that the portion of $\partial D^1_\uparrow$ parallel to $\partial \eta(T\cap H)$ and the portion of $\partial D^1_\downarrow$ parallel to $\partial\eta(T\cap H)$ are on opposite sides of $\partial\eta(T\cap H)$. See Figure 4. A small isotopy of $D^1_\downarrow$ will also guarantee that $D^2_\uparrow$ and $D^1_\downarrow$ are disjoint. By claim 1 we may assume that if $H = S^2$ then $|T\cap H| \neq 3$, and so one of $D^1_\uparrow$ or $D^2_\uparrow$ is a cut disk for $H-T$. It follows that if $D^1_\downarrow$ is a cut disk for $H-T$, then $H-T$ is c-weakly reducible contradicting our assumption. We conclude that $D^2_\downarrow$ is a cut disk for $H-T$. A small isotopy makes $D^1_\uparrow$ disjoint from $D^2_\downarrow$. Thus, if $D^1_\uparrow$ is a cut disk the disks $D^1_\uparrow$ and $D^2_\downarrow$ show that $H-T$ is c-weakly reducible, a contradiction. Thus, neither $D^1_\uparrow$ nor $D^1_\downarrow$ is a cut disk. Let $E_\uparrow$ and $E_\downarrow$ be the disks obtained by applying Lemma 8.3 to $D^1_\uparrow$ and $D^1_\downarrow$ respectively. By our choice of labelling for $D^1_\uparrow$ and $D^1_\downarrow$, the arcs $E_\uparrow\cap H$ and $E_\downarrow \cap H$ share a single endpoint. Hence, H is perturbed.

Case 5b: The endpoints of $\partial D_\uparrow$ on $\partial\eta(v)\subset H$ alternate with the endpoints of $\partial D_\downarrow$.

In this case, each of the pairs $\{D^j_\uparrow, D^k_\downarrow\}$ for $\{j,k\}\subset\{1,2\}$ is a pair of once-punctured disks with boundaries intersecting transversally exactly once and so H is bimeridionally stabilized.

□(Claim 2)

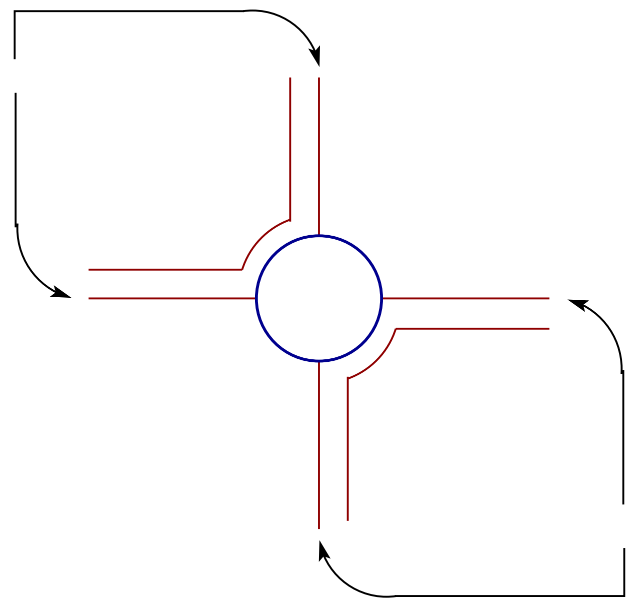

FIGURE 4. Choosing the labels $D^1_\uparrow$ and $D^1_\downarrow$

Interchange maxima and interchange minima of $h|_T$ as necessary to guarantee that $h|_F$ has critical values at $t_{\min}$ and $t_{\max}$. Let

$$t_{\min} = s_0 < s_1 < \ldots < s_n = t_{\max}$$

be the critical values of $h|_F$ between $t_{\min}$ and $t_{\max}$. Let $I_i = (s_i, s_{i+1})$ for $1 \leq i \leq n-1$. Label I_i with $\uparrow$ if for $t \in I_i$, there exists an upper disk for H_t in F and label I_i with $\downarrow$ if there exists a lower disk for H_t in F. By Claim 2, no I_i is labelled both $\uparrow$ and $\downarrow$. Furthermore, for no i is I_i labelled $\downarrow$ and I_{i+1} labelled $\uparrow$, since $H_{t_{i+1}}$ would have both an upper disk and a lower disk contained in F. Since $h|_F$ has critical values at $t_{\min}$ and $t_{\max}$ and since F is in normal form, I_0 is labelled $\downarrow$ and I_{n-1} is labelled $\uparrow$. Thus, there exists some i so that I_i is not labelled $\uparrow$ or $\downarrow$. Isotope H_t to eliminate arcs and circles of intersection which are inessential on both $H_t - \mathring{\eta}(T)$ and F. By our choice of $t_{\min}$ and $t_{\max}$, for $t \in I_i$, $H_t \cap F$ is non-empty. Since I_i is unlabelled, each arc and circle of intersection is essential in F. □(Theorem)

9. LEVELING EDGES

Suppose M is a closed manifold. A *Heegaard spine* for M is a graph $T \subset M$ such that the exterior of T in M is a handlebody. The *genus* of T is defined to be the genus of $\partial\eta(T)$. We say that T is *reducible* if there exists a sphere in M intersecting an edge of T transversally in a single point. If T is not reducible, it is *irreducible*.

In this section we will prove the following.

Theorem 9.1. *Suppose H is a Heegaard surface for M and T is an irreducible trivalent Heegaard spine for a closed manifold M in minimal bridge position with respect to H. Then one of the following occurs:*

1. *H is stabilized, meridionally stabilized, or bimeridionally stabilized as a splitting of (M,T).*
2. *T has a perturbed level edge.*
3. *T contains a perturbed level cycle.*
4. *There is an essential meridional surface F in the exterior of T such that* $\text{genus}(F) \leq \text{genus}(H)$.

This theorem is a partial generalization of the following result of Scharlemann and Thompson:

Theorem 9.2. [ST4] *Suppose that T is a trivalent genus 2 Heegaard spine for S^3. If T is isotoped to be in thin position with respect to a Heegaard sphere H for S^3 then T is in extended bridge position with respect to H and some interior edge of T is a perturbed level edge.*

We begin by proving a corollary of Theorem 9.1. For the proof we will need the following result of Morimoto:

Theorem 9.3. [Mo] *Suppose that T is a genus 2 Heegaard spine for a closed orientable 3–manifold M. Suppose that $S \subset M$ is a 2–sphere transverse to T such that $S \cap T$ is contained in non-separating edges of T. Then either M contains a lens space or $S^1 \times S^2$ connected summand or $S - \mathring{\eta}(T)$ is inessential in $M - \mathring{\eta}(T)$.*

Corollary 9.4. *Suppose that T is a genus 2 trivalent Heegaard spine for S^3 in minimal bridge position with respect to a Heegaard sphere H for S^3. Suppose that every edge of T is non-separating in T. Then some edge e of T can be isotoped to lie in H by a proper isotopy in $M - \mathring{\eta}(T - e)$.*

Proof. Since H is not stabilized as a splitting of S^3, it is neither stabilized nor meridionally stabilized as a splitting of (S^3, T). Since every edge of T is non-separating and $\text{genus}(T) \geq 2$, no edge transversally intersects a sphere in S^3 exactly once. By Morimoto's theorem, the exterior of T does not contain an essential meridional planar surface. Thus, by Theorem 9.1, T contains a perturbed level edge or a perturbed level cycle. Let e be either the perturbed level edge, or one of the edges contained in the perturbed level cycle. It is not difficult to use the perturbing disks to isotope e in $M - \mathring{\eta}(T - e)$ so that it lies in H. □

For the remainder we consider the following more general situation. Let T be a finite trivalent graph containing at least one edge embedded in a compact orientable 3–manifold M. For simplicity, assume that $T \cap \partial M = \emptyset$. Let H be a Heegaard surface for (M,T) dividing M into compression bodies C_1 and C_2.

We begin by considering the various methods of unperturbing Heegaard surfaces. Let $D_1 \subset C_1$ and $D_2 \subset C_2$ be disks which form a perturbing pair for H. Let $p = \partial D_1 \cap \partial D_2$.

The next lemma is essentially [STo2, Lemma 3.1].

Lemma 9.5. *If $\partial D_1 \cup \partial D_2$ is disjoint from the vertices of T, then there is an isotopy of T which reduces $|T \cap H|$ by two and which is supported in a regular neighborhood of $D_1 \cup D_2$.*

This lemma is stated, but not proved, in [HS1, Section 3].

Lemma 9.6. *Suppose that ∂D_1 contains one vertex v of T and that ∂D_2 does not contain any vertices of T. Let τ be the component of $T \cap C_2$ adjacent to $(\partial D_1 \cap (T \cap H)) - p$. If τ does not contain a vertex of T, then there is an isotopy of T supported in a neighborhood of $D_1 \cup D_2$ which reduces $|T \cap H|$.*

Proof. Choose a complete collection of bridge disks Δ for $T \cap C_2$ containing D_2. Out of all such collections, choose Δ to minimize $|\partial\Delta \cap (\partial D_1 \cap H)|$.

Claim: $\partial\Delta$ is disjoint from the interior of $\partial D_1 \cap H$.

Suppose not, and let x be the point of $\partial\Delta \cap \text{int}(\partial D_1 \cap H)$ closest to p. Let α be the path in $\partial D_1 \cap H$ from x to p. Let D be the disk of Δ containing x. The frontier in C_2 of a regular neighborhood of $D \cup \alpha \cup D_2$ has a disk component D' which is a bridge disk for $\partial D \cap T \cap C_2$. Replacing D with D' in Δ creates a collection which contradicts our choice of Δ. □(Claim)

Let τ' be the component of $T \cap C_1$ which contains v and let e_1 be the edge of τ' which lies in ∂D_1 but does not contain p. Let e be $(\partial D_1 \cup \partial D_2) - e_1$. Isotope τ' so that $e_2 = e \cap C_1$ is moved across D_1 into C_2. See Figure 5. Let S be the resulting parallelism between e_2 and H in C_2. By the Claim, $\Delta - D_2$ is disjoint from S. Thus, $(\Delta - D_2) \cup (D_2 \cup S)$ is a complete collection of bridge disks for $T \cap C_2$. Since τ does not contain a vertex of T, each component of $T \cap C_2$ is a pod or a bridge edge. Thus, H is still a Heegaard splitting of (M,T).

□

Corollary 9.7. *Suppose that $T \cap \partial M = \varnothing$ and suppose that T has been isotoped so that it is in minimal bridge position with respect to H. If T is perturbed, it contains a perturbed level edge.*

Proof. Suppose that H is perturbed by disks $D_1 \subset C_1$ and $D_2 \subset C_2$. By Lemma 9.5, one of ∂D_1 or ∂D_2 contains a vertex of T. If both do, then T has

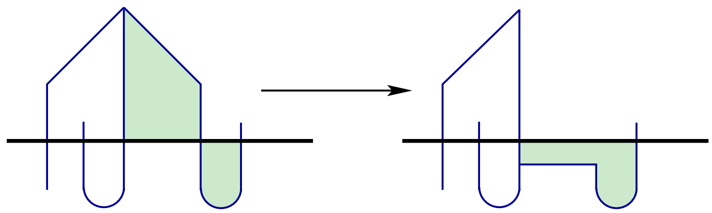

FIGURE 5. Unperturbing a Heegaard splitting

a perturbed level edge. Suppose, therefore, that only ∂D_1 contains a vertex v of T. Let τ be the component of $T \cap C_2$ adjacent to $(\partial D_1 \cap T \cap H) - p$. By Lemma 9.6, τ contains a vertex w of T. There is, therefore, an edge e joining v to w and intersecting H exactly once. Since $T \cap \partial M = \varnothing$, there is an edge $e' \subset C_2$ adjacent to w which is disjoint from $\partial D_1 \cup \partial D_2$. Let D' be a bridge disk for $e' \cup (e \cap C_2)$. By the proof of Lemma 9.6, we may assume that the arc $\partial D' \cap H$ has its interior disjoint from the arc $\partial D_1 \cap H$. Thus, (D_1, D') is a perturbing pair for T which shows that e is a perturbed level edge. □

Proof of Theorem 9.1. Let T be an irreducible trivalent Heegaard spine for the closed 3–manifold M. Let F be a complete collection of boundary reducing disks for the exterior of T. Since T is irreducible, no component of ∂F is a meridian of T.

Isotope T so that it is in minimal bridge position with respect to H. Assume that neither conclusion (1) nor conclusion (4) occurs. If $H - T$ is c-weakly reducible, by Theorem 7.2, either H is perturbed or T contains a perturbed level cycle. If $H - T$ is c-strongly irreducible in $M - T$, then by Theorem 8.1, one of the following occurs:

(a) H is perturbed
(b) T has a perturbed level cycle
(c) H can be isotoped to intersect F in a non-empty collection of arcs and simple closed curves all of which are essential in F.

Since F is the pairwise disjoint union of disks, (c) is impossible. If H is perturbed, by Corollary 9.7, T has a perturbed level edge. Thus, T has either a perturbed level edge or a perturbed level cycle. □

REFERENCES

[CG] A. Casson and C. Gordon. Reducing Heegaard splittings. *Topology Appl.*, 27(3):275–283, 1987.

[CM] S. Cho and D. McCullough. The tree of knot tunnels. *Geom. Topol.*, 13 (2009), no. 2, 769–815.

[C] A. Coward Algorithmically detecting the bridge number of hyperbolic knots. *arXiv:0710.1262*

[G] D. Gabai. Foliations and the topology of 3-manifolds. III. *J. Differential Geom.*, 26 (1987), no. 3, 479–536.

[GST] H. Goda, M. Scharlemann, and A. Thompson. Levelling an unknotting tunnel. *Geom. Topol.*, 4 (2000), 243–275 (electronic).

[GL] C. McA. Gordon, C. McA. and J. Luecke. Knots are determined by their complements. *J. Amer. Math. Soc.*, 2 (1989), no. 2, 371–415.

[H] W. Haken. Some results on surfaces in 3-manifolds. *Studies in Modern Topology*, 39–98, 1968.

[HS1] C. Hayashi and K. Shimokawa. Heegaard splittings of the pair of the solid torus and the core loop. *Rev. Mat. Complut.*, 14(2):479–501, 2001.

[HS2] C. Hayashi and K. Shimokawa. Heegaard splittings of trivial arcs in compression bodies. *J. Knot Theory Ramifications*, 10(1):71–87, 2001.

[HS3] C. Hayashi and K. Shimokawa. Thin position of a pair (3-manifold, 1-submanifold). *Pacific J. Math.*, 197(2):301–324, 2001.

[L] M. Lackenby. Heegaard splittings, the virtually Haken conjecture and property (τ). *Invent. Math.*, 164 (2006), no. 2, 317–359.

[Ma] J. Maher. Heegaard gradient and virtual fibers. *Geom. Topol.* 9 (2005), 2227–2259 (electronic).

[Mo] K. Morimoto, Planar surfaces in a handlebody and a theorem of Gordon-Reid, KNOTS '96 (Tokyo), 123–146, (1997)

[MS] Moriah, Yoav ; Sedgwick, Eric . Closed essential surfaces and weakly reducible Heegaard splittings in manifolds with boundary. *J. Knot Theory Ramifications*, 13 (2004), no. 6, 829–843.

[S] M. Scharlemann. Thin position in the theory of classical knots. *Handbook of knot theory*, 429–459, Elsevier B. V., Amsterdam, 2005.

[ST4] M. Scharlemann and A. Thompson. Thinning genus two Heegaard spines in S^3. *J. Knot Theory Ramifications*, 12 (2003), no. 5, 683–708

[ST3] M. Scharlemann and A. Thompson. Heegaard splittings of $(\text{surface}) \times I$ are standard. *Math. Ann.*, 295(3):549–564, 1993.

[ST2] M. Scharlemann and A. Thompson, Abigail. Thin position and Heegaard splittings of the 3-sphere. *J. Differential Geom.* 39 (1994), no. 2, 343–357.

[ST1] M. Scharlemann and A. Thompson. Thin position for 3-manifolds. In *Geometric topology (Haifa, 1992)*, volume 164 of *Contemp. Math.*, pages 231–238. Amer. Math. Soc., Providence, RI, 1994.

[STo2] M. Scharlemann and M. Tomova. Uniqueness of bridge surfaces for 2-bridge knots. *Mathematical Proceedings, Cambridge Philosophical Society*,

[STo1] M. Scharlemann and M. Tomova. Alternate Heegaard genus bounds distance. *Geom. Topol.*, 10 (2006), 593–617 (electronic).

[Sc] J. Schultens. The classification of Heegaard splittings for (compact orientable surface)$\times S^1$, *Proc. London Math. Soc. (3)*, vol. 67 (1993), 2:425–448

[TT] S. Taylor and M. Tomova. Heegaard surfaces for certain graphs in compression bodies. *arXiv:0910.3019*

[T2] M. Tomova. Multiple bridge surfaces restrict knot distance. *Algebr. Geom. Topol.*, 7 (2007), 957–1006.

[T1] M. Tomova. Thin position for knots in a 3-manifold. *J. London Mathematical Society*, (2) 80 (2009) 85–98.

[W] F. Waldhausen. Heegaard-Zerlegungen der 3-Sphäre. *Topology*, 7 (1968) 195–203.