ADDING 2–HANDLES TO SUTURED MANIFOLDS

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ABSTRACT. Combinatorial sutured manifold theory is used to compare the effects of attaching a 2–handle to essential simple closed curves on a genus two boundary component of a compact, orientable 3–manifold. The main theorems are applied to both arbitrary 2–handle attachment and to a limited form of 2–handle attachment known as "refilling meridians". Generalizations and new proofs of several well-known theorems from classical knot theory are obtained, including superadditivity of knot genus under band connect sum and the fact that unknotting number one knots are prime.

1. INTRODUCTION

Sutured manifold theory has produced many stunning results in the study of Dehn surgery on knots. Many of these insights are due to a famous theorem of Gabai:

Theorem (Gabai [G2, Corollary 2.4] (cf. [ScM2, Theorem 5.1]))**.** *Let M be a Haken manifold whose boundary is the non-empty union of tori. Let S be a Thurston norm minimizing surface representing an element of $H_2(M,\partial M)$ and let P be a component of ∂M such that $P \cap S = \varnothing$. Then with at most one exception (up to isotopy) S remains norm minimizing in each manifold obtained by Dehn filling M along a slope in P.*

For example, Lackenby [L1] used the ideas lying behind the proof of this theorem to study the intersections between surfaces and null-homotopic knots in 3–manifolds.

Adding a 2–handle to a boundary component of genus greater than 1 is a natural extension of Dehn surgery to higher genus boundary components, but sutured manifold theory has not often been exploited in this study; typically, 2–handle addition has been studied using purely combinatorial techniques. This paper applies the ideas lying behind Gabai's theorem and Lackenby's results to the study of 2–handle addition.

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The sutured manifold results are somewhat complicated to state precisely, so in this introduction we will content ourselves with stating applications that don't require the language of sutured manifold theory in their statement. All of the applications pertain to adding 2–handles to a genus 2 boundary component F of a 3–manifold N. This restriction to genus 2 allows us to maintain control over the sutures on F. Some of the theorem numbers in this introduction are marked with a prime. This indicates that the statement has been simplified (for reasons of exposition) by strengthing hypotheses or weaking the conclusion.

A 3–manifold N is **simple** if it does not contain an essential sphere, disc, annulus, or torus. For an essential simple closed curve $a \subset \partial N$, let $N[a]$ denote the result of attaching a 2–handle to a. If b is another simple closed curve on ∂N, such that a and b have been isotoped to intersect minimally, let $\Delta = \Delta(a,b)$ denote the number of points of intersection of a and b. The first application gives some evidence for a conjecture of Scharlemann and Wu:

Theorem 4.1'. *Suppose that N is a compact, orientable, simple 3–manifold, that $F \subset \partial N$ is a genus 2 component, and that $\partial N - F$ consists of tori. Suppose that a and b are non-isotopic separating curves on F. If $N[a]$ is reducible, then $N[b]$ is simple.*

The paper [T1] contains the only other application of sutured manifold theory to the study of 2–handle addition. It focused, almost entirely, on a limited form of 2-handle attachment, called "refilling meridians of a genus 2 handlebody". The other applications in this paper continue the project of that paper, although the sutured manifold techniques in this paper are considerably different and, in most cases, the results are stronger. It is not necessary to have prior knowledge of [T1].

Suppose that M is a compact, orientable 3-manifold containing a genus 2 handlebody $W \subset M$. Let α and β be essential discs in W which have been isotoped to intersect minimally. Let $N = M - \mathring{\eta}(W)$, $a = \partial\alpha$ and $b = \partial\beta$. We say that the 3–manifold $N[a]$ is obtained by **refilling** the meridian α. Similarly, $N[b]$ is obtained by refilling the meridian β. Let $\overline{\alpha}$ and $\overline{\beta}$ denote the cocores of the 2–handles $\eta(\alpha) \subset W$ and $\eta(\beta) \subset W$ respectively.

In [ScM4], Scharlemann conjectured conditions which would guarantee that at least one of the manifolds $N[a]$ or $N[b]$ obtained by refilling meridians α and β of W is irreducible and boundary-irreducible. Theorem 5.4 gives an almost complete solution to Scharlemann's conjecture.

Theorem 5.4'. *Suppose that W is a genus 2 handlebody embedded in a compact, orientable manifold M. Let α and β be essential discs in W which cannot be isotoped to be disjoint. Assume the following:*

- *any two curves of ∂M which compress in M are on the same component of ∂M.*
- *There is no essential sphere, disc, or annulus in $N = M - \mathring{W}$.*

Then all of the following hold:

- *One of $N[a]$ and $N[b]$ is irreducible and is not a solid torus*
- *If one of $N[a]$ or $N[b]$ is reducible then the other is boundary-irreducible*
- *If $c_a \subset \partial M$ is a curve which compresses in $N[a]$ and if $c_b \subset \partial M$ is a curve which compresses in $N[b]$ then c_a and c_b cannot be isotoped in ∂M to be disjoint.*

If α does not separate W, then the core loop of the solid torus $W - \mathring{\eta}(\alpha)$ is a knot in M. If α separates W, then the core loops of the solid tori $W - \mathring{\eta}(\beta)$ are a 2–component link in M. In either case, we let L_α denote the knot or link formed by the core(s) of $W - \mathring{\eta}(\alpha)$. Similarly, L_β denotes the knot or 2–component link formed by the core(s) of $W - \mathring{\eta}(\beta)$. If $\Delta(a,b) \geq 1$, then we say that L_β is obtained by **boring** L_α using handlebody W and **boring arc** $\overline{\alpha}$. Likewise, L_α is obtained by boring L_β using boring arc $\overline{\beta}$. Boring is simply "refilling meridians" phrased in terms of knots and links. The non-simplified version of Theorem 5.4 implies:

Corollary 5.6'. *Suppose that L_α is obtained by boring L_β using the handlebody $W \subset S^3$. If $S^3 - \mathring{\eta}(W)$ is boundary-irreducible, then one of L_α or L_β is neither a split link nor an unknot.*

In fact, a much stronger result, Theorem 5.5, is possible. This implies both the superadditivity of genus under band-sum and the fact that tunnel number one knots and links have minimal genus Seifert surfaces disjoint from a given tunnel.

Theorem 5.5. *Suppose that L_α is a knot or link in S^3 obtained by boring a split link or unknot L_β using a handlebody W. Then either*

(1) L_α is a split link and $\overline{\alpha}$ intersects a splitting sphere once, or
(2) L_α is not a split link and there is a minimal genus Seifert surface for L_α with interior disjoint from $\overline{\alpha}$.

In addition to studying the presence of essential spheres and discs and minimal genus Seifert surfaces in the exterior of a knot or link obtained by boring a split link or unknot, the techniques of this paper are useful for studying many other essential surfaces. The next theorem concerns the presence of essential annuli and tori in the exterior of a knot or link obtained by boring a split link.

Theorem 5.9'. *Suppose that L_β is a knot or link obtained by boring the split link L_α using a handlebody $W \subset S^3$. Assume that $N = S^3 - \mathring{W}$ is simple. If the exterior of L_β contains an essential annulus or torus then $\Delta = 2$.*

Rational tangle replacement is a particularly interesting type of boring. The next two theorems concern essential planar surfaces and (punctured) tori in the exterior of a knot or link obtained by a rational tangle replacement operation. The first is a generalization of, and gives a new proof of, the fact that unknotting number one knots are prime. See Section 5.4 for the relevant definitions.

Theorem 5.14. *Suppose that $L_\beta \subset S^3$ is a knot or link obtained by a rational tangle replacement of distance $d \geq 1$ on the knot or link L_α. Suppose that L_α is a split link or does not contain a minimal genus Seifert surface with interior disjoint from $\overline{\alpha}$. If L_α is a split link, also assume that $\overline{\alpha}$ does not intersect a splitting sphere just once.*

Suppose that $\overline{Q}$ is an essential properly embedded meridional planar surface in the exterior of L_β chosen so that out of all such surfaces $|\partial\overline{Q}|$ is minimal and, subject to that constraint, $|\overline{Q} \cap \overline{\beta}|$ is minimal. Then either $\overline{Q}$ is disjoint from $\overline{\beta}$ or

$$|\overline{Q} \cap \overline{\beta}|(d-1) \leq |\partial\overline{Q}| - 2.$$

There is also an interesting relationship between a Seifert surface for a knot or link obtained by rational tangle replacement on a split link or unknot and the distance of the rational tangle replacement. This theorem is directly related to Theorem 5.5.

Theorem 5.12. *Suppose that $L_\beta \subset S^3$ is a knot or link obtained by rational tangle replacement of distance $d \geq 1$ on a split link or unknot L_α. Assume that*

- *if L_α is a split link then $\overline{\alpha}$ does not intersect a splitting sphere just once*
- *if L_α is an unknot then $\overline{\alpha}$ is not disjoint from the interior of an unknotting disc for L_α,*

then L_β is not a split link or unknot and L_β has a minimal genus Seifert surface $\overline{Q}$ with interior disjoint from $\overline{\beta}$ so that at least one of the following holds:

(1) $\overline{\beta}$ *is isotopic into* $\overline{Q}$*.*
(2) $-\chi(\overline{Q}) \geq d$ *and* L_α *is a split link*
(3) $-\chi(\overline{Q}) \geq d-1$ *and* L_α *is an unknot.*

Finally, we present a theorem which may be useful for studying non-simple Dehn filling on simple knot or link exteriors. In the non-simplified version of the theorem, much more detail is given concerning the last possibility.

Theorem 5.16'. *Suppose that the hypotheses of the first paragraph of Theorem 5.14 hold.*

If there is a properly embedded essential planar surface or punctured torus $\overline{R}$ *in the exterior of* L_β*, then one of the following holds:*

(1) L_β *is a link and* $\partial\overline{R}$ *is disjoint from some component of* L_β*.*
(2) $\overline{R}$ *has meridional boundary on some component of* L_β
(3) The arc $\overline{\beta}$ *can be slid and isotoped to lie in* $\overline{R}$*.*
(4) $d \leq 3$*.*

Notation and Conventions. We assume a working knowledge of combinatorial sutured manifold theory as developed in [ScM2]. All (co)homology groups have integer coefficients. The term **link** will always mean a 2–component link.

S^3 denotes the 3–sphere, $\mathring{X}$ and int$X$ denote the interior of X, and $\eta(X)$ denotes a closed regular neighborhood of X. I denotes the interval $[0,1]$.

N will always denote a compact, orientable 3–manifold with boundary component F of genus at least 2. $N[a]$ denotes the 3–manifold obtained by attaching a 2–handle with core α to an essential simple closed curve $a \subset F$. The cocore of the 2–handle $\alpha \times I$ is denoted by $\overline{\alpha}$. Similarly b will always be an essential simple closed curve curve on F and β and $\overline{\beta}$ will be the core and co-core of a 2–handle attached to it. $\partial_1 N[a]$ and $\partial_1 N[b]$ both denote $\partial N - F$. $\partial_0 N[a]$ and $\partial_0 N[b]$ denote $\partial N[a] - \partial_1 N[a]$ and $\partial N[b] - \partial_1 N[b]$ respectively. M will be a compact orientable 3–manifold containing a genus two handlebody W. For essential discs α and β in W, L_α and L_β will denote the core of the solid torus/tori $W - \mathring{\eta}(\alpha)$ and $W - \mathring{\eta}(\beta)$ respectively. We will often tacitly assume that $\overline{\alpha}$ and $\overline{\beta}$ have been extended so that their endpoints lie on L_α and L_β. Similarly, we will not distinguish between a Seifert surface for a link L and the intersection of the Seifert surface with the exterior of the link.

A properly embedded surface in a 3-manifold is **essential** if every component is incompressible and not boundary parallel and if no component is a

2–sphere bounding a 3–ball or disc with null-homotopic boundary in the boundary of the 3–manifold.

Given a properly embedded surface $Q \subset N$, define an a**–boundary compressing disc** for Q to be a boundary compressing disc D for Q such that $\partial D = \delta \cup \varepsilon$ where δ and ε are arcs with $\partial\delta = \partial\varepsilon = \delta \cap \varepsilon$. The arc δ is an essential arc in Q. The arc ε is a subarc of some essential circle in $\eta(a) \subset F$.

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2. Sutured manifolds and 2–handle addition

In this section we state and then prove the main result of this paper: a theorem regarding the effect of attaching a 2–handle to the boundary of a taut sutured manifold. Sutured manifold theory was created by Gabai [G1] to construct foliations of 3–manifolds. Scharlemann produced a purely combinatorial version [ScM2]. It is that version we use here. The statement of the main result uses only the basic definitions from sutured manifold theory, but the proof will rely on the technical details of the construction of sutured manifold hierarchies.

2.1. Sutured Manifold Theory Terminology.

The following definitions can all be found in [ScM2]. They are, however, distributed throughout that paper and so we collect them here. The parenthetical numbers indicate the location in [ScM2] where the definition can be found.

Let (X,Γ,ψ) be a sutured manifold (2.1). X is an orientable 3–manifold, Γ is a collection of oriented simple closed curves on ∂X, and ψ is a (possibly empty) 1–complex properly embedded in X. The **sutures** in X are the disjoint union of tori $T(\Gamma)$ and annuli $A(\Gamma) = \eta(\Gamma)$ in ∂X. In this paper, $T(\Gamma)$ does not play a significant role, so, unless otherwise stated, $T(\Gamma) = \varnothing$. The surfaces $R_+(\Gamma)$ and $R_-(\Gamma)$ are the subsets of $\partial X - (T(\Gamma) \cup \mathring{A}(\Gamma))$ with outward and inward pointing normal vectors, respectively. We let $R(\Gamma) = R_+(\Gamma) \cup R_-(\Gamma)$.

A **conditioned surface** (2.4) $S \subset X$ is an oriented surface such that

(1) all components of ∂S on $T(\Gamma)$ are coherently oriented parallel essential circles,

(2) all components of ∂S in a component of $\eta(\Gamma)$ are either parallel coherently oriented circles or essential coherently oriented arcs,
(3) no collection of simple closed curves of $\partial S \cap R(\Gamma)$ is trivial in
$$H_1(R(\Gamma), \partial R(\Gamma)), \text{ and}$$
(4) an any edge of ψ which intersects $S \cup R(\Gamma)$ always does so with the same sign.

Suppose that $\overline{\psi} \subset \psi$ is an arc component and that D is a disc in X with interior disjoint from ψ such that $\partial D \cap \psi = \overline{\psi}$. If ∂D runs across $\overline{\psi}$ once, and if $|\partial D \cap \Gamma| = 1$, then D is a **cancelling disc** for $\overline{\psi}$. If ∂D runs twice across $\overline{\psi}$ and if ∂D is disjoint from Γ then D is a **self-amalgamating disc** for $\overline{\psi}$ (4.1).

An oriented surface $S \subset X$ is **ψ–taut** (1.2) in (X, Γ, ψ) if $S - \psi$ is incompressible in $X - \psi$, if any given edge of ψ always intersects S with the same sign, and if $\max\{0, |S \cap \psi| - \chi(S)\}$ is minimal among all embedded surfaces representing $[S, \partial S]$ in $H_2(X, \partial S)$. The sutured manifold (X, Γ, ψ) is **ψ–taut** (2.2) if $X - \psi$ is irreducible, if the ends of ψ are disjoint from $A(\Gamma) \cup T(\Gamma)$, and if $T(\Gamma)$ and $R(\Gamma)$ are ψ–taut. We usually write "taut" rather than $\varnothing$–taut. The quantity $\max\{0, -\chi(S)\}$ is the **Thurston norm** of S.

Suppose that Q is a surface properly embedded in $X - \mathring{\eta}(\psi)$. Q is a **parameterizing surface** (7.1) if no component of Q is a disc with boundary in $R_\pm(\Gamma)$. Suppose that $\psi = \varnothing$ and that the intersection of ∂Q with each component of $A(\Gamma)$, if non-empty, consists of essential arcs. The **index** $I(Q)$ of Q is the quantity
$$I(Q) = |\partial Q \cap \Gamma| - 2\chi(Q).$$
Certain sequences of sutured manifold decompositions can be modified so that they **respect** a parameterizing surface Q (7.7). If a sequence of sutured manifold decompositions respects Q, then after each decomposition we are left with a new parameterizing surface. The indices of the parameterizing surfaces do not increase under such a sequence of decompositions.

2.2. **The main results.** Recall that N is a compact, orientable 3-manifold and that $a \subset \partial N$ is an essential simple closed curve. Orient a and suppose that in ∂N a collection of pairwise disjoint oriented essential simple closed curves γ, all disjoint from the simple closed curve $a \subset F$, has been chosen so that $(N, \gamma \cup a)$ is a taut sutured manifold. Any tori of ∂N may be placed in either $T(\gamma \cup a)$ or in $R_\pm(\gamma \cup a)$. Notice that $(N[a], \gamma)$ is also a sutured manifold.

Main Theorem A. *Let $(N, \gamma \cup a)$ be a taut sutured manifold, as described above. Suppose that $Q \subset N$ is a compact orientable surface such that ∂Q intersects $a \cup \gamma$ minimally and no component of Q is a sphere or a disc disjoint from $\gamma \cup a$.*

If either of the following holds:

- $(N[a], \gamma)$ *is not taut, or*
- *There exists a surface S in $N[a]$ which is disjoint from $\overline{\alpha}$, is a conditioned surface in $N[a]$, and is taut in N but is not taut in $N[a]$,*

then one of the following holds:

(1) $N[a]$ contains an essential separating sphere intersecting $\overline{\alpha}$ exactly twice and which cannot be isotoped to intersect that arc any fewer times. Furthermore, this sphere bounds a non-trivial homology ball in $N[a]$.
(2) There is a compressing disc or a–boundary compressing disc for Q.
(3) $-2\chi(Q) \geq |\partial Q \cap a| - |\partial Q \cap \gamma|$.

The applications of Main Theorem A all compare the effects of two different 2–handle additions. Rather than apply Main Theorem A directly, we rephrase it so that instead of using a parameterizing surface $Q \subset N$, we use a surface $\overline{Q}$ lying in a manifold obtained by adding 2–handles to F. For such a surface $\overline{Q}$, we define a number called $K(\overline{Q})$ which will, in certain circumstances, give a lower bound on $-2\chi(\overline{Q})$. The number $K(\overline{Q})$ is simply a reworking of the index of a parameterizing surface.

Suppose that $\mathscr{B} = \{b_1, \ldots, b_{|\mathscr{B}|}\}$ is a collection of pairwise disjoint, pairwise nonparallel essential simple closed curves in $F \subset \partial N$ each of which intersects $a \cup \gamma$ minimally. Let $N[\mathscr{B}]$ be the result of attaching 2–handles to each of the curves in $\mathscr{B}$ and then capping off any 2–sphere boundary components with 3–balls. Let $\overline{Q} \subset N[\mathscr{B}]$ be an orientable surface and define $Q = \overline{Q} \cap N$. Assume that $\overline{Q}$ has been isotoped so that ∂Q intersects $\gamma \cup a$ minimally. Let q_i denote the number of components of ∂Q parallel to b_i. Additionally, define $\Delta_i = |b_i \cap a|$, $\Delta_\partial = |\partial \overline{Q} \cap a|$, $\nu_i = |b_i \cap \gamma|$, $\nu_\partial = |\partial \overline{Q} \cap \gamma|$, and

$$K(\overline{Q}) = (\Delta_\partial - \nu_\partial) + \sum_{i=1}^{|\mathscr{B}|} q_i(\Delta_i - \nu_i - 2).$$

Main Theorem B. *Suppose that N, γ, a, $\mathscr{B}$, and Q are as described above. Assume that no component of Q is a sphere or disc disjoint from $\gamma \cup a$.*

If either of the following holds:

- *$(N[a],\gamma)$ is not taut*
- *There exists a surface S in $N[a]$ which is disjoint from $\overline{\alpha}$, is a conditioned surface in $N[a]$, and is taut in N but is not taut in $N[a]$,*

then one of the following holds:

(1) $N[a]$ contains an essential separating sphere intersecting $\overline{\alpha}$ exactly twice and which cannot be isotoped to intersect that arc any fewer times. Furthermore, this sphere bounds a non-trivial homology ball in $N[a]$
(2) There is a compressing disc or a–boundary compressing disc for Q
(3) $-2\chi(\overline{Q}) \geq K(\overline{Q})$.

Proof of Main Theorem B given Main Theorem A. Notice that the statements of Main Theorem B and Main Theorem A differ only in the last possible conclusion. We show, therefore, that the final conclusion of Main Theorem A implies the final conclusion of Main Theorem B.

Assume that
$$-2\chi(Q) \geq |\partial Q \cap a| - |\partial Q \cap \gamma|.$$
Since Q is obtained by removing $\sum q_i$ discs from $\overline{Q}$, we have:
$$\chi(Q) = \chi(\overline{Q}) - \sum q_i.$$
For each i, q_i of these discs have boundary parallel to b_i. Thus,
$$|\partial Q \cap \gamma| = \nu_\partial + \sum q_i \nu_i$$
and
$$|\partial Q \cap a| = \Delta_\partial + \sum q_i \Delta_i.$$
Combining these equations with our assumed inequality, we obtain:
$$-2(\chi(\overline{Q}) - \sum q_i) \geq (\Delta_\partial + \sum q_i \Delta_i) - (\nu_\partial + \sum q_i \nu_i).$$
Rearranging this inequality produces $-2\chi(\overline{Q}) \geq K(\overline{Q})$. □

The idea behind the proof of Main Theorem A is to take a taut sutured manifold hierarchy of $(N,\gamma\cup a)$ which is "disjoint" from a, attach a 2–handle to a at the end of the hierarchy, and, noticing that the result must be non-taut, perform a simple combinatorial argument. However, in order to apply the main theorems of sutured manifold theory, we need several preliminary results concerning sutured manifold hierarchies. These theorems are stated for a sutured manifold $(X,\Gamma\cup a,\psi)$ rather than the sutured manifold $(N,\gamma\cup a)$ since they do not use the hypotheses of Main Theorem A. The proofs of these results are quite technical, the first-time reader may wish to skim them.

2.3. **Constructing a sutured manifold hierarchy.** Let $(X, \Gamma \cup a, \psi)$ be a sutured manifold. X is a 3–manifold, a is a simple closed curve on ∂X which is one of the sutures, and ψ is a 1–complex properly embedded in X. Let $\overline{\alpha}$ be the cocore of a 2–handle attached to ∂X along a. The arc $\overline{\alpha}$ is oriented so that it runs from $R_-(\Gamma)$ to $R_+(\Gamma)$ in the sutured manifold $(X[a], \Gamma, \psi \cup \overline{\alpha})$. We say that the sutured manifold $(X, \Gamma \cup a, \psi)$ is obtained by **converting** the arc $\overline{\alpha}$ in the sutured manifold $(X[a], \Gamma, \psi \cup \overline{\alpha})$ into a suture. Similarly, we say that $(X[a], \Gamma, \psi \cup \overline{\alpha})$ is obtained by converting the suture a into an arc. Converting an arc to a suture or *vice versa* does not affect tautness:

Lemma 2.1 ([ScM3, Lemma 2.3])**.** *The sutured manifold $(X[a], \Gamma, \psi \cup \overline{\alpha})$ is $(\psi \cup \overline{\alpha})$–taut if and only if $(X, \Gamma \cup a, \psi)$ is ψ–taut.*

The next lemma will provide the surfaces that are essential for creating a useful sutured manifold hierarchy.

Lemma 2.2. *Suppose that $(X, \Gamma \cup a)$ is a taut sutured manifold and that $H_2(X[a], \partial X[a]) \neq 0$. Then, in $X[a]$, there is a conditioned surface S which is disjoint from $\overline{\alpha}$. Furthermore, S is a taut conditioned surface in X, $\partial S \cap \eta(a) = \varnothing$, and no closed component of S separates.*

Proof. By the proofs of Theorems 2.5 and 2.6 of [ScM2], given a non-trivial homology class $z \in H_2(X[a], \partial X[a])$ there exists a taut conditioned surface S' in the $\overline{\alpha}$–taut sutured manifold $(X[a], \Gamma, \overline{\alpha})$. After possibly replacing z with $-z$ we may assume that $\overline{\alpha}$ has algebraic intersection number $i \leq 0$ with S'. By the choice of orientation of the arc $\overline{\alpha}$, $\overline{\alpha}$ has algebraic intersection number $+1$ with $R_-(\Gamma)$. The surface S'' which is the double curve sum of S' with i copies of $R_-(\Gamma)$ has algebraic intersection number zero with $\overline{\alpha}$. Notice that if a surface in X has boundary $\partial S''$ then it is conditioned. Tube together points of $S'' \cap \overline{\alpha}$ with opposite intersection number to create from S'' a surface S which is disjoint from $\overline{\alpha}$ and for which $\partial S = \partial S''$. The surface S is a conditioned surface in $(X, \Gamma \cup a)$. After replacing S with a taut surface in $(X, \Gamma \cup a)$ having the same boundary, we have constructed the desired surface. Since S is taut any closed component separating component must be a 2–sphere or torus. We may discard such components without changing the homology class of S or its Thurston norm. □

In constructing a taut sutured manifold decomposition that respects a parameterizing surface Q, it may be necessary to take the double curve sum of a conditioned surface S with some number k of copies of R_+ and some number l of copies of R_-, creating the surface $S_{k,l}$. We then decompose using the surface $S_{k,l}$ instead of S. See [ScM2] for more details on this construction and its necessity.

Suppose that S is a conditioned surface in $(X[a],\Gamma)$ which is disjoint from $\overline{\alpha}$. (Such a surface might be provided by Lemma 2.2.) The surface $S_{k,l}$ may have boundary components which are located in $\eta(a)$. After a small isotopy, attaching discs to each of those boundary components creates a surface $S^a_{k,l} \subset X[a]$. The surface $S^a_{k,l}$ can also be created by taking the double curve sum of S with k copies of $R_+(\Gamma) \subset X[a]$ and l copies of $R_-(\Gamma) \subset X[a]$.

The next lemma guarantees that if we use such a surface to perform a decomposition of the sutured manifold $(X[a],\Gamma,\overline{\alpha})$ then all but one arc of $\overline{\alpha} - \mathring{\eta}(S^a_{k,l})$ can be cancelled. Parameterize the 2-handle attached to a as $D^2 \times I$, with D^2 the unit disc in the complex plane. Let $*_a$ be the point on $\overline{\alpha}$ corresponding to $(0,1/2) \in D^2 \times I$. Under the standard deformation retraction of $D^2 \times I$ to $\overline{\alpha}$, the curve a is retracted to $*_a$. We will always assume that, when forming $S^a_{k,l}$, the k copies of R_+ intersect $\overline{\alpha} = \{0\} \times I$ in points $(0,p)$ for $p > 1/2$. Likewise, we always assume that the l copies of R_- intersect $\overline{\alpha}$ in points $(0,p)$ for $p < 1/2$.

Suppose that $(X[a],\Gamma,\psi \cup \overline{\alpha})$ is a sutured manifold and that the sutured manifold $(X'[a],\Gamma',\psi' \cup \overline{\alpha}')$ is obtained by decomposing $X[a]$ along a surface Σ which is disjoint from $*_a$. The notation means that $\psi' = \psi - \mathring{\eta}(\Sigma)$ and $\overline{\alpha}' = \overline{\alpha} - \mathring{\eta}(\Sigma)$. Define a **residual arc** to be a component of $\overline{\alpha}'$ which does not contain $*_a$. The next lemma shows that residual arcs may be cancelled.

Lemma 2.3. *Let $S^a_{k,l} \subset X[a]$ be a surface created from a conditioned surface S which is disjoint from $\overline{\alpha}$. Then, for any k and l, after decomposing $X[a]$ along $S^a_{k,l}$ there exists a cancelling disc or a self-amalgamating disc for each residual arc. Furthermore, these cancelling discs and self-amalgamating discs may be taken to be pairwise disjoint.*

Proof. Let $(X,\Gamma \cup a,\psi)$ be a sutured manifold and suppose that S is a conditioned surface in $X[a]$. For $k,l \in \mathbb{N} \cup \{0\}$, let $S^a_{k,l}$ be the double curve sum of S with k copies of $R_+(\Gamma) \subset X[a]$ and l copies of $R_-(\Gamma) \subset X[a]$. Notice that $S^a_{k,l}$ is disjoint from $*_a$. Let $(X[a]',\Gamma',\psi' \cup \overline{\alpha}')$ be the sutured manifold obtained by decomposing $X[a]$ along $S^a_{k,l}$. Recall that $\psi' = \psi - \mathring{\eta}(S^a_{k,l})$ and $\overline{\alpha}' = \overline{\alpha} - \mathring{\eta}(S^a_{k,l})$.

Consider the union of arcs A which is the closure of $\overline{\alpha} - S^a_{k,l}$ in $X[a]$. Since S is disjoint from $\overline{\alpha}$, each component of A has endpoints which lie on $R_+(\Gamma) \cup R_-(\Gamma)$ or on one of the k parallel copies of $R_+(\Gamma)$ or on one of the l parallel copies of $R_-(\Gamma)$. There is at most one component of A with an endpoint on $R_+(\Gamma)$ and at most one component of A with an endpoint on $R_-(\Gamma)$ since such arcs have, as one of their endpoints, an endpoint of $\overline{\alpha}$. By the construction of $S^a_{k,l}$ the only component of A which joins a copy of $R_-(\Gamma)$ (possibly, $R_-(\Gamma)$ itself) to a copy of $R_+(\Gamma)$ (possibly, $R_+(\Gamma)$ itself) is the component of A which contains $*_a$. All other components of A are contained in a product region between two copies of $R_-(\Gamma)$ or between two copies of $R_+(\Gamma)$. Indeed, in this product region such a component of A is a vertical arc. In $X[a]'$, the components of $\overline{\alpha}$ are obtained by removing a regular neighborhood of the the endpoints of the components of A. Thus, each residual arc of $\overline{\alpha}'$ is a vertical arc in a product sutured manifold.

Let $\delta \subset \overline{\alpha}'$ be a residual arc in $X[a]'$. Let d be the component of A which contains δ, so that δ is obtained by removing a regular neighborhood of ∂d. Let P be the product region between two copies of $R_-(\Gamma)$ or two copies of $R_+(\Gamma)$ which contains d. Parameterize P as $R_\pm(\Gamma) \times I$. The arc d is $x \times I$ for some $x \in R_\pm(\Gamma)$. Let $R_i = R_\pm(\Gamma) \times \{i\} \subset P$ for $i \in \{0,1\}$.

Case 1: The component of $R_0 - S$ containing x intersects ∂R_0.

In this case, let p be an embedded arc in $R_0 - S$ joining x to ∂R_0. Then $p \times I \subset P$ is a rectangle in P which has one edge on d. The disc $(p \times I) \cap X[a]'$ is then a cancelling disc for δ.

Case 2: The component of $R_0 - S$ containing x does not intersect ∂R_0.

Since S is conditioned, no component of $R_0 - S$ is a disc with boundary $R_0 \cap S$. Thus, in this case, it is possible to choose an essential closed curve p in $R_0 - S$ which passes through x. Let $D = p \times I \subset P$. The disc $D \cap X[a]'$ is a self-amalgamating disc for δ.

Notice from the construction that it is possible to take the collection of cancelling discs and self-amalgamating discs for residual arcs to be disjoint. □

Let $(X, \Gamma \cup a, \psi)$ be a ψ–taut sutured manifold. Suppose that $S \subset X$ is ψ–taut conditioned surface which is disjoint from $\eta(a)$. Let $S_{k,l}$ be the double curve sum of S with k copies of $R_+(\Gamma)$ and l copies of $R_-(\Gamma)$. Let $(X', \Gamma' \cup a, \psi')$ be the ψ'–taut sutured manifold obtained by decomposing along $S_{k,l}$. The sutured manifold X' will have sutures arising from the boundary components of $S_{k,l}$ in $\eta(a)$. Call such sutures **residual sutures**.

Corollary 2.4. *After the decomposition* $(X, \Gamma \cup a, \psi) \stackrel{S_{k,l}}{\to} (X', \Gamma' \cup a, \psi')$, *there is a product disc in* X' *for each residual suture.*

Proof. The decomposition can be achieved by converting the suture a into an arc $\overline{\alpha}$, decomposing along $S^a_{k,l}$ and then converting each residual arc and the component of $\overline{\alpha} - \mathring{\eta}(S^a_{k,l})$ back into sutures. The cancelling discs and self-amalgamating discs for the residual arcs become the desired product discs in X'. □

When creating a sutured manifold hierarchy, we will need to carefully handle residual arcs and sutures. Here is a method for doing this called **careful cancellation**.

Lemma 2.5. *Let (X,Γ,ψ) be a ψ–taut sutured manifold and let $\delta \subset \psi$ be an arc component. Suppose that D is a cancelling disc or self-amalgamating disc for δ. If D is a self-amalgamating disc, assume that the arcs $\partial D \cap R(\Gamma)$ are essential in $R(\Gamma) - \psi$. Then there is a ψ–taut decomposition $(X,\Gamma,\psi) \to (X',\Gamma',\psi')$. The sutured manifold X' has a component B which is a 3–ball with a single suture in its boundary and which contains the arc δ. The sutured manifold $X' - B$ can be obtained from X by cancelling the arc δ using D or decomposing along a product annulus determined by D.*

Proof. First, suppose that D is a cancelling disc. A regular neighborhood of D is a 3–ball B containing δ. Decomposing X along the disc $E = \text{cl}(\partial B \cap \mathring{X})$ produces a sutured manifold, one component of which is the ball B which now has a single suture in its boundary. The other components of the decomposed sutured manifold can be obtained simply by cancelling the arc δ in X. By [ScM2, Lemma 4.2] the decomposed sutured manifold is taut.

Convert the arc δ in the decomposed sutured manifold to a suture. Let (X',Γ',ψ') be the resulting sutured manifold. Converting δ into a suture transforms B into a solid torus T with two longitudinal sutures on its boundary. Thus, (X',Γ',ψ') is a ψ'- taut sutured manifold.

Now suppose that D is a self-amalgamating disc for δ. Pushing D off δ creates a non-trivial product annulus A in X. The hypothesis on the essentiality of $\partial D \cap R(\Gamma)$ guarantees that ∂A is essential in $R(\Gamma) - (\psi - \delta)$.

There is a parallelism of δ into A. Decompose X along A and notice that there is now a cancelling disc D' for δ in this new sutured manifold. Decomposing along a non-trivial product annulus preserves tautness by [ScM2, Lemma 4.2]. Use D' to cancel δ as above creating (X',Γ',ψ'), a ψ'- taut sutured manifold. One component of X' is a 3–ball B with a single suture in its boundary containing δ. □

Carefully cancelling a residual arc in a sutured manifold produces a 3–ball with a single suture in its boundary containing the residual arc. Call the 3–ball a **residual ball**. Notice that a residual ball is $\varnothing$-taut.

Here is the version for cancelling residual sutures.

Lemma 2.6. *Suppose that*

$$(X,\Gamma\cup a,\psi)\xrightarrow{S_{k,l}}(X',\Gamma'\cup a,\psi')$$

is a ψ–taut decomposition as in Corollary 2.4. Let D be a product disc for a residual suture $d\subset\Gamma'$. Then there is a ψ–taut sutured manifold decomposition $(X',\Gamma'\cup a,\psi')\to(X'',\Gamma''\cup a,\psi')$. The sutured manifold X'' has a component T which is a solid torus with two longitudinal sutures on its boundary, one of which is d. The sutured manifold $X''-T$ can be obtained from X' by decomposing along the product disc D.

Proof. Convert the suture a into an arc. The disc D is then a cancelling disc or self-amalgamating disc for the residual arc δ corresponding to d. Carefully cancel the arc δ as in Lemma 2.5. Let B be the residual ball containing the arc δ. Let X'' be the result of converting all the remnants of $\overline{\alpha}$ back into sutures. This converts the ball B into a solid torus with two longitudinal sutures on its boundary, one of which is d. The other components of X'' can be obtained from X' by decomposing along the product disc D since prior to the arc to suture conversion they could be obtained by cancelling the arc δ or decomposing along a product annulus. □

The solid torus T in the previous lemma is called a **residual torus**.

We now assemble these lemmas to construct certain hierarchies of sutured manifolds.

Proposition 2.7. *Suppose that $(X,\Gamma\cup a)$ is a taut sutured manifold. Let Q be a parameterizing surface in X. Then there exists a sequence of taut sutured manifold decompositions*

$$(\dagger)\quad (X,\Gamma\cup a)=(X_0,\Gamma_0\cup a)\xrightarrow{S_1}(X_1,\Gamma_1\cup a,\overline{\alpha}_1)\xrightarrow{S_2}\dots\xrightarrow{S_n}(X_n,\Gamma_n\cup a,\overline{\alpha}_n)$$

respecting the parameterizing surface Q such that

(1) Each decomposition occurs in one of the following ways:
 (a) it is a decomposition along a surface $S_{k,l}$ given by Lemma 2.2.
 (b) it is a careful cancellation of a suture
 (c) it is a decomposition along a product disc disjoint from a.
(2) Every residual suture is cancelled.
(3) No residual torus is ever decomposed.

(4) *Let T denote the residual tori components of X_n. Then $H_2(X_n[a] - T, \partial X_n[a] - T) = 0$.*
(5) *If a component of X_n does not contain a then it is either a residual torus or a 3–ball with a single suture in its boundary.*

Proof. This is essentially the proof that taut sutured manifold hierarchies exist [ScM2, Theorems 4.19 and 7.8]. If $H_2(X_0[a], \partial X_0[a]) = \varnothing$ then the theorem is trivially true. Suppose, therefore that the sequence has been extended to a taut sutured manifold $(X_i, \Gamma_i \cup a)$ in such a way that each decomposition satisfies (1). By Corollary 2.4, there is a product disc crossing each residual suture. Thus, by Lemma 2.6, we may also assume that all residual sutures have been carefully cancelled. Decompose along any other product discs which are disjoint from a and the residual sutures. (This is the "eliminate index-zero discs" step at the beginning of [ScM2, Theorem 4.19].) If there is a product disc crossing a, carefully cancel a. If $H_2(X_i[a] - T, \partial X_i[a] - T) = 0$, then each component of X_n not containing a is either a residual torus or has spherical boundary. Since the decompositions are taut each such component with spherical boundary is a 3–ball with a single suture in its boundary. That is, Conclusion (5) holds.

Thus, we may assume that $H_2(X_i[a] - T, \partial X_i[a] - T) \neq 0$. By Lemma 2.2 there is a taut surface S in X_i which is disjoint from $\eta(a)$. Decomposing X_i along the surface $S_{k,l}$ is a taut sutured manifold decomposition which respects Q for large enough k,l ([ScM2, Lemma 7.5]). By Corollary 2.4 and Lemma 2.6 we may carefully cancel all residual sutures in the decomposed manifold and then eliminate all other product discs. Thus, we are able to extend our sequence of decompositions to $(X_{i+m}, \Gamma_{i+m} \cup a)$ for some $m > 0$.

The proof of [ScM2, Theorem 4.19] guarantees that this process cannot be continued indefinitely. □

Proposition 2.8. *Suppose that $(X, \Gamma \cup a)$ is a $\varnothing$–taut sutured manifold. Let Q be a parameterizing surface in X. Then there exists a sequence of sutured manifold decompositions*

$$(\dagger\dagger) \qquad (X[a], \Gamma, \overline{\alpha}) = (X_0^a, \Gamma_0, \overline{\alpha}_0) \xrightarrow{S_1^a} (X_1^a, \Gamma_1, \overline{\alpha}_1) \xrightarrow{S_2^a} \dots \xrightarrow{S_n^a} (X_n^a, \Gamma_n, \overline{\alpha}_n)$$

respecting the parameterizing surface Q such that

(1) *each decomposition occurs in one of the following three ways:*
 (a) *it is a decomposition along a surface $S_{k,l}^a$ obtained from a surface $S_{k,l}$ given by Lemma 2.2.*
 (b) *it is a careful cancellation of an arc component using a cancelling disc*

(c) it is a careful cancellation of an arc component using self-amalgamating disc satisfying the hypotheses of Lemma 2.5.
(d) it is a decomposition along a product disc

(2) Every residual arc is cancelled.
(3) No residual ball is ever decomposed.
(4) $H_2(X^a_n, \partial X^a_n) = 0$
(5) if a component of X^a_n *is disjoint from* $\overline{\alpha}_n$ *then it is a 3–ball with a single suture in its boundary.*

Proof. In the hierarchy (†) provided by Proposition 2.7 change each surface S_i into a surface S^a_i as follows:

- If S_i is a surface $S_{k,l}$ provided by Lemma 2.2, let $S^a_i = S^a_{k,l}$.
- If S_i is a product disc which crosses a residual suture once, let S^a_i be the corresponding cancelling disc.
- If S_i is a product disc which crosses a residual arc just once, let S^a_i be the corresponding self amalgamating disc.
- If S_i is a product disc disjoint from a, let $S^a_i = S_i$. Note that by the construction of (†), S_i does not cross a residual suture.

Since a careful cancellation of a product disc or self amalgamating disc corresponds by construction to a careful cancellation of a residual suture, the sequence (††) has all of the desired properties by Proposition 2.7. □

Remark. The challenge in proving Main Theorem A is to correctly apply the theorems of sutured manifold theory. The route that we have taken is to use Proposition 2.7 to guarantee that the sequence of decompositions of $(N, \gamma \cup a)$ terminates. This sequence is then converted into a sequence of decompositions of the sutured manifold $(N[a], \gamma, \overline{\alpha})$. Forgetting the arc $\overline{\alpha}$ then gives a sequence of sutured manifold decompositions of $(N[a], \gamma)$ to which [ScM2, Corollary 3.9] can be applied. This is done in the following corollary. We continue to phrase the results in terms of $(X, \Gamma \cup a)$ as we do not yet need the hypotheses pertaining to N. We use the notation from the previous theorem.

Let $\overline{\alpha}_*$ be the arc component of $\overline{\alpha}_n$ which contains $*_a$. Let X_* be the component of X^a_n which contains $\overline{\alpha}_*$.

Corollary 2.9. *If* $(X_*, \Gamma_n \cap X_*)$ *is* $\varnothing$*–taut then* $(X[a], \Gamma)$ *and the surface* S^a_1 *are* $\varnothing$*–taut.*

Proof. Recall the hierarchy (††). Since all arcs of $\overline{\alpha}_n - \overline{\alpha}_*$ are contained in residual balls, the hypotheses of the lemma are equivalent to the claim that if (X^a_n, Γ_n) is $\varnothing$–taut then $(X[a], \Gamma)$ is $\varnothing$–taut.

Consider the sequence of sutured manifold decompositions

$$(X[a],\Gamma) = (X_0^a,\Gamma_0) \xrightarrow{S_1^a} (X_1^a,\Gamma_1) \xrightarrow{S_2^a} \dots \xrightarrow{S_n^a} (X_n^a,\Gamma_n)$$

obtained from the hierarchy (††) by forgetting the arc $\overline{\alpha} \subset X[a]$ and all its remnants in each X_i^a. Each surface S_i^a along which a decomposition is performed is either a conditioned surface, a product disc, or an essential product annulus with both endpoints essential in $R_\pm(\Gamma_{i-1})$. (These qualities all follow from the construction of the cancelling and self amalgamating discs in Lemma 2.3 and the process of careful cancellation.) No component of $X[a]$ is a solid torus disjoint from Γ since $(X,\Gamma\cup a)$ is $\varnothing$–taut. Also, no closed component of any S_i^a separates by the construction of the surfaces S_i. By [ScM2, Corollary 3.9], if (X_n^a,Γ_n) is $\varnothing$–taut, so is $(X_0^a,\Gamma_0) = (X[a],\Gamma)$. The surface S_1^a is also $\varnothing$–taut by [ScM2, Corollaries 3.3 and 3.9]. □

2.4. **The proof of Main Theorem A.** Recall that $(N,\gamma\cup a)$ is a taut sutured manifold and that either (N,γ) is not taut or there is a surface $S\subset N[a]$ which is disjoint from $\overline{\alpha}$ and taut in $(N,\gamma\cup a)$ but not taut in $(N[a],\gamma)$. Also recall that $Q\subset N$ is a parameterizing surface.

By choosing $X = N$ and $\Gamma = \gamma$ we can apply Proposition 2.7 to obtain a sequence of sutured manifold decompositions (†) of $(N,\gamma\cup a)$ respecting Q. It is permissible to choose the surface S_1 to be the double curve sum of the surface S in the hypotheses of Main Theorem A with k copies of $R_+(\gamma\cup a)$ and l copies of $R_-(\gamma\cup a)$. From the sequence of decompositions (†), Proposition 2.8 constructs a sequence of decompositions

$$(\dagger\dagger)\qquad (N[a],\gamma,\overline{\alpha}) = (N_0^a,\gamma_0,\overline{\alpha}_0) \xrightarrow{S_1^a} (N_1^a,\gamma_1,\overline{\alpha}_1) \xrightarrow{S_2^a} \dots \xrightarrow{S_n^a} (N_n^a,\gamma_n,\overline{\alpha}_n).$$

The surface S_1^a can be taken to be $S_{k,l}^a$ with S the surface provided by the hypotheses of Main Theorem A and $k,l \in \mathbb{N}\cup\{0\}$.

Let N_* be the component of N_n^a containing the point $*_a$.

Lemma 2.10. *$(N_*,\gamma_n\cap N_*)$ is not $\varnothing$-taut.*

Proof. Suppose, to the contrary, that $(N_*,\gamma_n\cap N_*)$ is $\varnothing$–taut. By Corollary 2.9, both $(N[a],\gamma)$ and the surface $S_{k,l}^a$ are taut. If $S_{k,l}^a$ is taut in $(N[a],\gamma)$, then so is S. To see this, recall that the $S_{k,l}^a$ is obtained by taking the double curve sum of S with k copies of $R_+(\gamma)$ and l copies of $R_-(\gamma)$. If S is not $\varnothing$–taut, then it does not minimize the Thurston norm (in $H_2(N[a],\partial S)$). But in this case, the double curve sum of S with k copies of R_+ and l copies of R_- is not Thurston norm minimizing either, implying that $S_{k,l}^a$ is not taut, a contradiction.

Thus, either $(N[a],\gamma)$ is taut or S is taut in $(N[a],\gamma)$. This contradicts the hypotheses of Main Theorem A. □

Carefully examining N_* will enable us to conclude the proof of the theorem. Rather than considering N_* directly, it will facilitate the exposition if we instead consider $(N',\gamma'\cup a)$ which obtained from $(N_*,\gamma_n\cap N_*,\overline{\alpha}_*)$ by converting $\overline{\alpha}_*$ to a suture a. (Notice that $\gamma' = \gamma_n \cap N_*$.)

Lemma 2.11. *$\partial N'$ is a torus and $N'[a] = N_*$ is an integer homology ball.*

Proof. The proof is similar to [L2, Lemma A.4]. Let $A = \partial N' - \mathring{\eta}(a)$. By construction of the hierarchy, $H_2(N',A) = 0$. Thus, by duality for manifolds with boundary, $H^1(N',\eta(a)) = 0$. By the Universal Coefficient Theorem, $H_1(N',\eta(a)) = 0$. From the exact sequence for the homology of the pair $(N',\eta(a))$, $H_1(\eta(a))$ surjects onto $H_1(N')$. Thus, $H_1(N')$ is cyclic. Since $H_2(N',A) = 0$, by the long exact sequence for the pair (N',A), $H_1(A)$ injects into $H_1(N')$. Since A is a surface and $\partial\eta(a)$ has two components, A is a collection of spheres and either an annulus or two discs. Since a does not compress in N, A does not contain a disc. (A compression of a would imply that $(N,\gamma\cup a)$ was not taut.) The existence of a sphere would contradict tautness of $(N,\gamma\cup a)$, and so A is an annulus. This implies that $\partial N'$ is a torus.

Since $H_1(A)$ is isomorphic to $\mathbb{Z}$ and it injects into the cyclic group $H_1(N')$, $H_1(N')$ is also isomorphic to $\mathbb{Z}$. Since $\eta(a)$ is an annulus and since $H_1(\eta(a))$ surjects $H_1(N')$, the inclusion of $\eta(a)$ into N' induces an isomorphism on first homology. Since A is an annulus and $H_2(N',A) = 0$, the exact sequence for the pair (N',A) shows that $H_2(N') = 0$. It is then easy to see that $N'[a]$ is a homology ball. □

We now resume the proof of Main Theorem A. Since $(N'[a],\gamma')$ is a sutured manifold and $\partial N'$ is a torus containing the suture a there must be an odd number r of other sutures. The proof of the theorem concludes by examining two cases. The first case is when $r = 1$ and the second case is when $r \geq 3$.

Suppose that $r = 1$. Then $\partial N'[a]$ is a sphere containing a single suture. Since $(N'[a],\gamma')$ is an integer homology ball (Lemma 2.11) which is not taut, the integer homology ball is not a 3–ball. Push $\partial N'[a]$ slightly into $N[a]$. Then, $\partial N'[a]$ must be a reducing sphere for $N[a]$ which is intersected exactly twice by $\overline{\alpha}$ and which bounds a non-trivial integer homology ball. If $\overline{\alpha}$ could be isotoped to intersect the sphere $\partial N'[a]$ fewer times, it could be isotoped to be disjoint from that sphere and N' would be reducible, contrary to the hypothesis that $(N,\gamma\cup a)$ is taut. Hence, conclusion (1) holds.

Suppose, therefore, that $r \geq 3$. The hierarchy (††) respects the parameterizing surface Q. Thus, at the end of the hierarchy, we have a parameterizing surface Q_n. Since no component of Q was a sphere or disc disjoint from $\gamma \cup a$, no component of Q_n is a sphere or disc disjoint from $\gamma \cup a$. The index of a parameterizing surface does not increase under a hierarchy. Thus, the index $I(Q)$ of Q is greater than or equal to the index $I(Q_n)$ of $Q_n \subset N_n^a$. Since no component of Q_n is a disc or sphere disjoint from $\gamma_n \cup \overline{\alpha}_n$, no component of Q_n has negative index. Let $Q' = Q_n \cap N'$. Hence, $I(Q) \geq I(Q')$.

Lemma 2.12. *Suppose that $\partial Q' \cap (\partial N' - \mathring{\eta}(a))$ contains an arc inessential in $\partial N' - \mathring{\eta}(a)$. Then one of the following occurs:*

(1) there is an isotopy of Q reducing $|\partial Q \cap a|$
(2) An outermost such arc bounds a disc in $\partial N' - \mathring{\eta}(a)$ which contains a compressing disc for Q in N.
(3) an outermost such arc in $\partial N' - \mathring{\eta}(a)$ bounds an a–boundary compressing disc D for Q in N.

Proof. This is a standard innermost disc/outermost arc argument. □

The first conclusion of Lemma 2.12 contradicts our hypothesis on Q. The second and third conclusions of 2.12 produce the second possible conclusion of Main Theorem A. Thus, we may assume that no arc of $\partial Q' \cap (\partial N' - \mathring{\eta}(a))$ contains an arc inessential in $\partial N' - \mathring{\eta}(a)$.

Each loop of $\partial Q' \cap \partial N'$ is an essential circle on the torus $\partial N'$. Let ζ be one such circle. Then ζ is a loop in $\partial N'$ which intersects a minimally a positive number of times. Hence, ζ intersects all $r+1$ sutures $\gamma' \subset \partial N'$.

Let Q' be a component of Q_n such that at least one component of $\partial Q'$ intersects a. Notice that $-2\chi(Q') \geq -2$. Let $z_{Q'} = |\partial Q' \cap a|$. Then $\partial Q'$ has at least $z_{Q'}(r+1)$ intersections with the sutures γ'. Hence,

$$I(Q') \geq z_{Q'}(r+1) - 2\chi(Q') \geq z_{Q'}(r+1) - 2 \geq z_{Q'}(r-1).$$

Then,

$$I(Q_n) \geq \sum I(Q') \geq (r-1) \sum z_{Q'}$$

where the sums are taken over all components Q' of Q_n which have at least one boundary component intersecting a. By the construction of Q_n from Q, we have that $\sum z_{Q'} = |\partial Q \cap a|$. Thus, by the definition of index,

$$|\partial Q \cap \gamma| + |\partial Q \cap a| - 2\chi(Q) = I(Q) \geq I(Q_n) \geq (r-1)|\partial Q \cap a|.$$

Consequently,

$$|\partial Q \cap \gamma| - 2\chi(Q) \geq (r-2)|\partial Q \cap a| \geq |\partial Q \cap a|$$

since $r \geq 3$. Hence,

$$-2\chi(Q) \geq |\partial Q \cap a| - |\partial Q \cap \gamma|$$

as desired. □

3. SUTURES AND a–BOUNDARY COMPRESSIONS

For Main Theorems A and B to be useful, we must place sutures γ on ∂N so that $(N, \gamma \cup a)$ is taut. We now describe how to do this, beginning with the sutures $\hat{\gamma}$ on F. For the remainder of the paper we require F to have genus two.

If a is separating, define $\hat{\gamma} = \varnothing$. If a is non-separating, define $\hat{\gamma}$ to be a pair of essential disjoint simple closed curves on $F - \mathring{\eta}(a)$ which separate $\partial \eta(a)$. The next lemma shows how to usefully define sutures $\widetilde{\gamma}$ on non-torus components of $\partial N - F$. Let $\gamma = \hat{\gamma} \cup \widetilde{\gamma}$.

Lemma 3.1 ([T1, Lemma 4.1])**.** *Suppose that $F - (\widehat{\gamma} \cup a)$ is incompressible in N and that N is irreducible. Suppose also that if $\partial N - F$ contains a non-torus component then there is no essential annulus in N with boundary on $\hat{\gamma} \cup a$. Then $\widetilde{\gamma}$ can be chosen so that $(N, \gamma \cup a)$ is taut. Furthermore, if $c \subset \partial N - F$ is a collection of disjoint, non-parallel curves such that:*

- $|c| \leq 2$
- *All components of c are on the same component of $\partial N - F$*
- *No curve of c cobounds an essential annulus in N with a curve of $\hat{\gamma} \cup a$*
- *If $|c| = 2$ then there is no essential annulus in N with boundary c*
- *If $|c| = 2$ and a is separating, there is no essential thrice-punctured sphere in N with boundary $c \cup a$.*

then $\widetilde{\gamma}$ can be chosen to be disjoint from c.

Remark. The statement of the lemma does not explicitly say what to do with torus components of ∂N. The torus boundary components can be placed in $T(\gamma \cup a)$ or put into $R_{\pm}(\gamma \cup a)$ without affecting the tautness of $(N, \gamma \cup a)$.

A surface $\overline{Q}$ plays an important role in Main Theorem B. The remainder of this section discusses the elimination of a–boundary compressions for such a surface.

Suppose that $b \subset F$ is a simple closed curve. If b is non-separating, there are multiple ways to obtain a manifold homeomorphic to $N[b]$. Certainly attaching a 2–handle to b is one such way. If b^* is any curve in F which cobounds in F with $\partial\eta(b)$ a thrice-punctured sphere, then attaching 2–handles to both b^* and b creates a manifold with a spherical boundary component. Filling in that sphere with a 3–ball creates a manifold homeomorphic to $N[b]$. We will often think of $N[b]$ as obtained in this fashion. If b is separating, define $b^* = \varnothing$. When we apply Main Theorem B we will choose $\mathscr{B} = \{b, b^*\}$, if $b^* \neq \varnothing$, and $\mathscr{B} = \{b\}$, if $b^* = \varnothing$.

Let $\overline{Q} \subset N[b]$ be an embedded surface and, as usual, let $Q = \overline{Q} \cap N$. The surface $\overline{Q}$ is **suitably embedded** if each component of $\partial Q - \partial\overline{Q}$ is a curve on F parallel to b or to some b^*. We denote the number of components of $\partial Q - \partial\overline{Q}$ parallel to b by $q = q(\overline{Q})$ and the number parallel to b^* by $q^* = q^*(\overline{Q})$. Let $\widehat{q} = q + q^*$.

To get the most use out of Main Theorem B, we would like to know that there are useful incompressible and *a*–boundary incompressible surfaces. Theorem 5.1 of [T1] allows us to find some. We state here the version of the theorem which will be useful for our applications.

Theorem 3.2.

(I) Suppose that $N[b]$ contains a non-empty collection $\overline{R}$ of essential discs and spheres. Then there is a suitably embedded essential disc or sphere $\overline{Q} \subset N[b]$ such that $\widehat{q}(\overline{Q}) \leq \widehat{q}(\overline{R})$ and
- *If $\overline{R}$ is a sphere or disc with boundary on $\partial_0 N[b]$ then $\overline{Q}$ is a sphere or disc with boundary on $\partial_0 N[b]$*
- $\partial\overline{Q} \cap \partial_1 N[b] \subset \partial\overline{R}$
- *There is no compressing disc or a–boundary compressing disc for $Q = \overline{Q} \cap N$.*

(II) Suppose that $N[b]$ does not contain an essential sphere or disc and that $\overline{R} \subset N[b]$ is an essential surface. Then there is a suitably embedded essential connected surface $\overline{Q} \subset N[b]$ such that the following hold:
- $-\chi(\overline{Q}) \leq -\chi(\overline{R})$ *and the genus of $\overline{Q}$ is no more than the total of the genera of the components of $\overline{R}$.*
- $\widehat{q}(\overline{Q}) \leq \widehat{q}(\overline{R})$
- *If $\overline{R}$ is non-separating, so is $\overline{Q}$.*
- $\partial\overline{Q} \cap \partial_1 N[b] \subset \partial\overline{R}$.
- *There is no compressing disc for Q and either $\overline{Q}$ is contained in N or there is no a–boundary compressing disc for Q.*

Proof. This observation easily follows from Theorem 5.1 and Corollary 5.2 of [T1], noting that in this paper we use $\widehat{q}$ for the quantity $\widetilde{q}$ in [T1]. The only point that is not immediate is how to guarantee that $\widehat{q}(\overline{Q}) \leq \widehat{q}(\overline{R})$ in case (II). The statement of the theorem and corollary only guarantee that

$(-\chi(\overline{Q}), \widehat{q}(\overline{Q})) \leq (-\chi(\overline{R}), \widehat{q}(\overline{R}))$ in lexicographic order. To achieve the stronger statement here, notice that under the operations described in the proof of [T1, Theorem 5.1] the only time $\widehat{q}$ increases is when (B1) of that theorem is invoked or when a–torsion $2g$–gons (with $g \geq 2$) are being eliminated. For the theorem here do not invoke (B1) and do not attempt to eliminate a–torsion $2g$–gons with $g \geq 2$. □

It is worth noting that the surface $\overline{Q}$ is created from $\overline{R}$ by compressing and boundary compressing $R = \overline{R} \cap N$ using a–boundary compressing discs. This observation will be useful later. We now turn to applications.

4. 2–Handle Addition

If N is a simple 3–manifold and $b \subset \partial N$ is an essential simple closed curve, b is called **degenerating** if $N[b]$ is non-simple. Degenerating handle additions producing reducible and boundary-reducible manifolds have been studied by several authors. Here is what is known. None of these results use sutured manifold theory.

Theorem. *Suppose that N is a simple 3–manifold and that $F \subset \partial N$ is a component of genus at least 2 containing essential simple closed curves a and b. Then*

- *If $N[a]$ is reducible and $N[b]$ is boundary-reducible then either $\Delta = 0$ or a and b can be isotoped to lie in a common once-punctured torus in ∂N* [SW, Theorem 4.2].
- *If $N[a]$ and $N[b]$ are both reducible, then $\Delta \leq 4$* [ZQL].
- *If* $\text{genus}(F) = 2$*, if a and b are both separating, and if $N[a]$ and $N[b]$ are both boundary-reducible, then $\Delta = 0$* [LQZ].

A degenerating curve $b \subset \partial N$ is called **basic** if b is separating or if there is no degenerating separating curve b^* bounding in ∂N a once-punctured torus containing b. Scharlemann and Wu conjecture:

Conjecture ([SW, Conjecture 2])**.** If a and b are basic degenerating curves on ∂N then $\Delta \leq 5$.

The next theorem gives some evidence for their conjecture. Its proof is logically independent from the prior results on 2-handle addition.

Theorem 4.1. *Suppose that F is a genus 2 boundary component of a simple 3–manifold N. Suppose that $a \subset F$ is an essential separating curve and that $b \subset F$ is a basic degenerating curve not isotopic to a. Suppose that $N[a]$ is reducible. Then one of the following is true:*

- *The curve b is non-separating and $\Delta \leq 2$.*
- *$\Delta \leq 4$ and $N[b]$ contains an essential annulus with both boundary components on non-torus components of $\partial N[b]$. Furthermore, if b is separating, $\Delta = 4$.*

Proof. Notice, first, that if $\widehat{b} \subset F$ is a separating curve not isotopic to a then $\Delta(a,\widehat{b}) \geq 2$. If $\Delta(a,\widehat{b}) = 2$ then $a - \widehat{b}$ would have a single arc on each once-punctured torus component of $F - \widehat{b}$, implying that a was non-separating. Thus, $\Delta(a,\widehat{b}) \geq 4$. We will apply this observation with $\widehat{b} = b$ if b is separating or with $\widehat{b} = b^*$ if b is non-separating.

Now suppose that $N[b]$ contains a surface $\overline{R}$ which is an essential sphere, disc, torus, or annulus. Theorem 3.2 guarantees the existence of a surface $\overline{Q}$ satisfying the conclusions of that theorem. Each component of $\partial Q - \partial \overline{Q}$ is parallel either to b or to a curve b^*. Out of all such surfaces, choose $\overline{Q}$ so that $\widehat{q}(\overline{Q})$ is minimal. Let $\Delta^* = |b^* \cap a|$.

If $N[b]$ contains an essential disc or sphere, then by (I) of Theorem 3.2, $\overline{Q}$ is an essential disc or sphere. If $N[b]$ does not contain an essential sphere or disc, then $\overline{Q}$ is an essential torus or annulus since $-\chi(\overline{Q}) \leq -\chi(\overline{R})$ and $\text{genus}(\overline{Q}) \leq \text{genus}(\overline{R})$.

Since N is simple, $\overline{Q}$ is not contained in N. Thus, by Theorem 3.2, there is no compressing disc or a–boundary compressing disc for Q and $\widehat{q}(\overline{Q}) > 0$. Recall that if b is separating $\widehat{q} = q$. If b is non-separating, then since it is basic, b^* is not degenerating and so $q > 0$.

Case 1: $\overline{Q}$ is not an annulus with both boundary components on non-torus components of $\partial_1 N[b]$.

Let $\widehat{\gamma} = \varnothing$. Since N is simple, N is irreducible, and $F - (\widehat{\gamma} \cup a)$ is incompressible in N. Let $c = \partial \overline{Q} \cap \partial_1 N$. By the hypothesis of this case and the fact that $\overline{Q}$ is a sphere, disc, annulus, or torus, $|c| \leq 1$. Since N is simple, there is no annulus in N with boundary equal to $c \cup a$. Thus, the hypotheses of Lemma 3.1 are satisfied. Hence, there are sutures $\widetilde{\gamma}$ on the non-torus components of $\partial N - F$ so that $(N, \gamma \cup a)$ is a taut sutured manifold. (Since $\widehat{\gamma} = \varnothing$, $\gamma = \widetilde{\gamma}$.) Furthermore, c is disjoint from the sutures $\gamma \cup a$.

By Lemma 3.1, $(N, \gamma \cup a)$ is a taut sutured manifold. Q is a properly embedded connected surface in N. Since $\overline{Q}$ was chosen to minimize $\widehat{q}(\overline{Q})$, ∂Q intersects $\gamma \cup a$ minimally. Indeed, $\partial Q \cap \gamma = \varnothing$, since c is disjoint from the sutures and $\widehat{\gamma} = \varnothing$. Since $q(\overline{Q}) > 0$, Q has at least one boundary component parallel to b. Since $\Delta > 0$, Q is not disjoint from a. By hypothesis, $N[a]$ is reducible and so $(N[a],\gamma)$ is not taut. Thus we may apply Main Theorem B with $\mathscr{B} = \{b, b^*\}$.

Since N does not contain an essential annulus, conclusion (1) does not occur. By the construction of $\overline{Q}$, conclusion (2) does not occur. Thus, conclusion (3) occurs.

Recall that since c is disjoint from the sutures $\widetilde{\gamma}$ and since $\widehat{\gamma} = \varnothing$, $|\partial Q \cap \gamma| = 0$. Hence,

$$K(\overline{Q}) = \Delta_\partial + q(\Delta - 2) + q^*(\Delta^* - 2).$$

Recall that $q > 0$ and $\Delta_\partial \geq 0$. If $q^* \neq 0$, then by the observations at the beginning of this proof, $\Delta^* \geq 4$. Hence,

$$K(\overline{Q}) \geq \Delta - 2.$$

Plugging this into the inequality from conclusion (3) of Main Theorem B we obtain:

$$-2\chi(\overline{Q}) \geq \Delta - 2$$

Since $\overline{Q}$ is a sphere, disc, annulus, or torus $0 \geq -2\chi(\overline{Q})$. Hence,

$$2 \geq \Delta.$$

If b was separating, then $\Delta \geq 4$ and so this case cannot occur.

Remark. The reason that the proof of Case 1 does not extend to the case when $\overline{Q}$ is an annulus with both boundary components on non-torus components of $\partial N[b]$ arises from the use of Lemma 3.1. The inequalities in Case 1 rely on the fact that the sutures $\widetilde{\gamma}$ can be chosen to be disjoint from $\partial\overline{Q}$.

Case 2: $\overline{Q}$ is an annulus with both boundary components on non-torus components of $\partial_1 N[b]$.

Let G be the components of $\partial_1 N$ containing $\partial\overline{Q}$. Let N' be the manifold obtained by doubling N along G. That is, N' is formed by gluing a copy N_2 of N to $N_1 = N$ along G. Let F_i, a_i, b_i, b_i^* Q_i be the copies of F, a, b, b_i^* and Q lying in N_i. The gluing should be performed so that $\overline{Q}' = \overline{Q} \cup Q_2$ is a punctured torus in $N'[b_1]$ with punctures on F_2, each of which is parallel to b_2 or b_2^*. It is easy to show that N' is simple. Notice that $N'[a_1]$ is reducible.

Let $Q' = \overline{Q'} \cap N'$. We need to know that Q' is incompressible and a_1–boundary incompressible. For brevity, we show only that Q' is a_1–boundary incompressible. Suppose that D is an a_1–boundary compressing disc for Q' with $\varepsilon = \partial D \cap F_1$. Since N_1 and N_2 are simple, we may assume that $D \cap G$ consists of arcs which are essential in G. Since there is no a–boundary compressing disc for Q in N, this collection of arcs is non-empty. Since G

is disjoint from F_1, there is some arc of $D \cap G$ which is outermost on D and does not contain ε in the outermost disc it bounds. Let E be the outermost disc bounded by that arc. Then E is a boundary compressing disc for Q_1 or Q_2. Without loss of generality, suppose it to be Q_1. Since $\overline{Q}$ is essential in $N_1[b_1]$, the arc $\partial E \cap \overline{Q}$ must be inessential in $\overline{Q}$. Cutting $\overline{Q}$ along ∂E produces a surface with an annulus component and a disc component. Since $N_1[b_1]$ contains no essential discs, the disc component must be inessential. But this implies that there is an isotopy of $\overline{Q}$ reducing $\widehat{q}$, contradicting our choice of $\overline{Q}$. Hence, there is no a_1–boundary compressing disc for Q'. The proof that Q' is incompressible is similar and easier.

Let $\widehat{\gamma} = \varnothing$. If $b^* = \varnothing$, let $c = b_2$; otherwise, let $c = b_2^*$. As in Case 1, we may use Lemma 3.1 to construct sutures $\widetilde{\gamma}$ on $\partial N'$ which are disjoint from c so that $(N', \gamma \cup a_1)$ is a taut sutured manifold. If $b^* = \varnothing$, then since $c = b_2$, $\partial\overline{Q}$ is disjoint from γ. If $b^* \neq \varnothing$, then $c = b_2^*$. The curve b_2^* is a separating curve on a genus 2 boundary component of $\partial N'$, thus b_2 is also disjoint from γ. Thus, both when $c = b_2^*$ and when $c = b_2$, $\partial\overline{Q}$ is disjoint from $\gamma \cup a$. Consequently,

$$K(\overline{Q}') = q(\Delta - 2) + q^*(\Delta^* - 2).$$

The surface $\overline{Q}'$ is a punctured torus with $\widehat{q}$ boundary components. Thus, $-2\chi(\overline{Q}') = 2\widehat{q}$. As in Case 1, we may apply Main Theorem B to conclude that $-2\chi(\overline{Q}') \geq K(\overline{Q}')$. That is,

$$2(q + q^*) \geq q(\Delta - 2) + q^*(\Delta^* - 2).$$

Rearrange this to obtain:

$$0 \geq q(\Delta - 4) + q^*(\Delta^* - 4).$$

If $q^* \neq 0$, then $\Delta^* \geq 4$ since b^* is separating. Also, $q > 0$. Thus, $0 \geq \Delta - 4$. Hence, $4 \geq \Delta$, as desired. If b is separating, then $\Delta \geq 4$ and so $\Delta = 4$. □

5. Refilling Meridians

5.1. **Preliminary Remarks.** For the remainder, suppose that W is a genus two handlebody embedded in a compact orientable 3–manifold M and take $N = M - \mathring{W}$. Assume that N is irreducible. Suppose that a and b are essential simple closed curves in $F = \partial W$ bounding discs α and β in W intersecting minimally and non-trivially. Recall that if b is non-separating then b^* may be any curve which cobounds with $\partial\eta(b) \subset F$ a thrice-punctured disc. Such a curve bounds an essential disc β^* in W.

We say that $N[a]$ is obtained by **refilling** the meridian α [ScM4]. The solid torus or tori $W - \mathring{\eta}(\alpha)$ is the regular neighborhood of a knot or 2-component link L_α in M. We say that L_α is obtained by **boring** L_β (and vice versa). The cocores $\overline{\alpha}$ and $\overline{\beta}$ of $\eta(\alpha)$ and $\eta(\beta)$ in W are called the **boring arcs**.

We will need an understanding of how a and b lie on F. The proofs of the following assertions are straightforward. More detail can be found in [ScM4] and [T1]. Each arc of $\alpha \cap \beta$ which is outermost on β bounds a subdisc of β which is a meridian of a solid torus component of $W - \mathring{\eta}(\alpha)$. The arc lying in b which forms part of the boundary of such a subdisc of β has both endpoints lying on the same component of $\partial\eta(a) \subset F$. Call an arc in b lying in $F - \mathring{\eta}(a)$ which, together with an arc in α, bounds a meridian disc of $W - \mathring{\eta}(\alpha)$ a **meridional arc** of $b - a$. Let $\mathscr{M}_a(b)$ denote the total number of meridional arcs of $b - a$. If a is non-separating then any arc of $b - \mathring{\eta}(a)$ with endpoints on the same component of $\partial\eta(a)$ is a meridional arc. In this case, there are the same number of meridional arcs based at each component of $\partial\eta(a)$. Thus, when a is non-separating, $\mathscr{M}_a(b)$ is even.

The next lemma will be useful later:

Lemma 5.1. *Suppose that $\overline{Q} \subset N[b]$ has $\partial Q \cap a = \varnothing$. Then, $\overline{Q} \subset N$ and $\partial\overline{Q}$ is either disjoint from or consists of meridians on some component of $\partial_0 N[b]$.*

Proof. Since $\Delta > 0$ and $\Delta^* > 0$, Q has no boundary components parallel to b or b^*. Hence, $\overline{Q} = Q$ and, therefore, $\overline{Q}$ lies in N. $\partial\overline{Q}$ is disjoint from the meridional arcs of $a - b$. Since there exist such meridional arcs lying on at least one component of $\partial_0 N[b]$, $\partial\overline{Q}$ is either disjoint from that component or consists of meridians on that component. □

We desire to choose sutures γ so that $(N, \gamma \cup a)$ is a taut sutured manifold. We begin by defining the sutures $\widehat{\gamma} \subset F$ as in Section 3. If a is separating, choose $\widehat{\gamma} = \varnothing$, as before. Otherwise, in the boundary of the solid torus $W - \mathring{\eta}(a)$ choose $\widehat{\gamma}$ to be two meridians of W separating the components of $\partial\eta(a)$, intersecting b minimally, and disjoint from the meridional arcs of $b - a$. The remaining sutures $\widetilde{\gamma}$ on $\partial N - F$ will (eventually) be constructed using Lemma 3.1. Assume, for the time being, that $\widetilde{\gamma}$ has been defined so that $(N, \gamma \cup a)$ is a sutured manifold (with $\gamma = \widetilde{\gamma} \cup \widehat{\gamma}$).

We will make frequent reference to the number $K(\overline{Q})$ for a suitably embedded surface $\overline{Q} \subset N[b]$. Recall that $\Delta = |a \cap b|$ and define $\nu = |b \cap \gamma|$, $\Delta^* = |b^* \cap a|$, $\nu^* = |b^* \cap \gamma|$. Let q be the number of components of $\partial Q - \partial\overline{Q}$ parallel to b and q^* be the number parallel to b^*. Then

$$K(\overline{Q}) = (\Delta - \nu - 2)q + (\Delta^* - \nu^* - 2)q^* + \Delta_\partial - \nu_\partial.$$

The next lemma shows why it is easier to prove theorems about refilling meridians than about arbitrary 2–handle attachments.

Lemma 5.2. *If $\partial\overline{Q}$ is disjoint from $\widetilde{\gamma}$ then $K(\overline{Q}) \geq 0$.*

Proof. If α is separating, then $\nu = \nu^* = 0$ since $\widehat{\gamma} = \varnothing$. Since, in addition, $\partial\overline{Q}$ is disjoint from $\widetilde{\gamma}$, $\nu_\partial = 0$. Thus, if α is separating,

$$K(\overline{Q}) = (\Delta - 2)q + (\Delta^* - 2)q^* + \Delta_\partial.$$

Since $|b \cap a| = 2|\beta \cap \alpha|$ and since $|b^* \cap a| = 2|\beta^* \cap \alpha|$, $K(\overline{Q})$ is non-negative if α is separating.

Suppose that α is non-separating. Then $\Delta - \nu = \mathscr{M}_a(b) \geq 2$. If in addition, $b^* \neq \varnothing$, then $\Delta^* - \nu^* \geq \mathscr{M}_a(b^*) \geq 2$. Since $\partial\overline{Q}$ is disjoint from b, in particular from the meridional arcs of $b - a$,

$$|\partial\overline{Q} \cap a| - |\partial\overline{Q} \cap \widehat{\gamma}| \geq 0.$$

Hence, if $\partial\overline{Q}$ is disjoint from $\widetilde{\gamma}$, $K(\overline{Q})$ is non-negative. □

Our final preliminary lemma concerns the existence of taut conditioned Seifert surfaces. It is a hands-on version of Lemma 2.2.

Lemma 5.3. *Suppose that L_α is null-homologous in M. Then there is a Seifert surface S for L_α which is disjoint from $\overline{\alpha}$ (i.e. lies in N) and is a taut conditioned surface in N with boundary disjoint from a.*

Proof. First we show that L_α does contain a conditioned Seifert surface disjoint from $\overline{\alpha}$. Choose a Seifert surface $\Sigma_0 \subset N[a]$ for L_α. Since L_α is null-homologous in M such a surface exists. If L_α is a link, Σ_0 may not be connected. Since $\partial\Sigma_0$ is longitudinal on $\partial N[a]$, when α is non-separating we may assume that $|\partial\Sigma_0 \cap \widehat{\gamma}| = 2$. The surface Σ_0 is disjoint from $\widetilde{\gamma}$ (whatever the choice of $\widetilde{\gamma}$).

Orient Σ_0 and recall that $\overline{\alpha}$ is oriented from $R_-(\gamma)$ to $R_+(\gamma)$. Calculate the algebraic intersection number between $\overline{\alpha}$ and each component of Σ_0. If it is $n \neq 0$, an endpoint of $\overline{\alpha}$ may be isotoped around $\partial N[a]$ creating n intersections of sign $-n/|n|$. Perform the isotopy so that $\partial\overline{\alpha}$ is always disjoint from γ. Rather than isotoping $\overline{\alpha}$, we may instead isotope Σ_0. We take this latter viewpoint. The requirement, from the former viewpoint, that $\partial\overline{\alpha}$ be disjoint from γ guarantees that, from the latter viewpoint, if α is non-separating then $\partial\Sigma_0$ still intersects each component of γ exactly once.

We may, therefore, assume that the intersection number of $\overline{\alpha}$ with each component of Σ_0 is zero. Choosing an arc σ of $\overline{\alpha} - \Sigma_0$ with endpoints creating intersections of opposite sign on the same component of Σ_0, we attach a tube containing σ to Σ_0, decreasing $|\Sigma_0 \cap \overline{\alpha}|$ (but increasing the genus of Σ_0). The algebraic intersection number of $\overline{\alpha}$ and Σ_0 is still zero. Continuing in this manner, we may construct a conditioned Seifert surface Σ for L_α which is disjoint from $\overline{\alpha}$. Out of all Seifert surfaces for L_α which are disjoint from $\overline{\alpha}$ and which have boundary $\partial\Sigma$ choose one of minimal Thurston norm and call it S. Then S is a taut surface in N. Because ∂S intersects each component of γ at most once, S is conditioned. □

Remark. Notice that even though ∂S (where S is the surface created by the previous lemma) is a longitude on $\partial_0 N[a]$ (when α is separating) it may intersect meridional arcs of $b - a$ more than once. It must, however, intersect them at least once. See Figure 1 for a depiction of the "spiralling $\partial\overline{\alpha}$" viewpoint. The boundary of S may not intersect b minimally (as indicated by the figure). Since S will act as a decomposing surface, what matters is that it is conditioned; we do not need it to intersect b or b^* minimally.

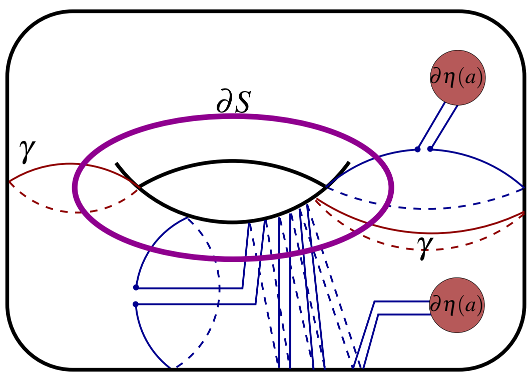

FIGURE 1. The result of spiralling $\partial\overline{\alpha}$ around $\partial_0 N[a]$

5.2. **Scharlemann's Conjecture.** In [ScM4], Scharlemann considered refilling meridians of a genus two handlebody W in a wide variety of compact, orientable 3–manifolds M. He showed in a number of situations that if both $N[a]$ and $N[b]$ are reducible or boundary-irreducible (with minor assumptions on the embedding of W in M) then either $M = S^3$ and W is unknotted or α and β are "aligned" in W. He conjectured that this is always the case (given his hypotheses on the pair (M,W)). In [T1], under slightly different hypotheses, significant progress was made on the conjecture. One case not covered by [T1], was the case when both $N[a]$ and $N[b]$ are solid tori. The next theorem shows that, under very mild hypotheses, if $\Delta > 0$, it is impossible for both $N[a]$ and $N[b]$ to be solid tori.

The conclusions of Theorem 5.4 are actually much stronger than those predicted by Scharlemann's conjecture. Our hypotheses are neither stronger nor weaker than those of Scharlemann's conjecture. That they are not stronger than those of Scharlemann's conjecture explains why portions of his conjecture remain open. After Theorem 5.4 and [T1], the only remaining major open case of Scharlemann's conjecture is the case when there is an annulus in N with one boundary component on a non-torus component of ∂M and one the boundary of a meridian of W. Minor differences between the statement of Scharlemann's conjecture, Theorem 6.1 in [T1], and Theorem 5.4 below also need to be straightened out.

Theorem 5.4. *Suppose that W is a genus 2 handlebody embedded in a compact, orientable manifold M. Let α and β be essential discs in W which cannot be isotoped to be disjoint. Assume the following:*

1. *(1) any two curves of ∂M which compress in M are on the same component of ∂M.*
2. *(2) If P is an essential sphere in M intersected transversally by $L_\alpha \cup \overline{\alpha}$ then $|P \cap (L_\alpha \cup \overline{\alpha})| \geq 3$. Similarly, if P is intersected transversally by $L_\beta \cup \overline{\beta}$ then $|P \cap (L_\beta \cup \overline{\beta})| \geq 3$.*
3. *(3) If P is an essential disc in M intersected transversally by $L_\alpha \cup \overline{\alpha}$ then $|P \cap (L_\alpha \cup \overline{\alpha})| \geq 2$. Similarly, if P is intersected transversally by $L_\beta \cup \overline{\beta}$ then $|P \cap (L_\beta \cup \overline{\beta})| \geq 2$.*
4. *(4) $N = M - \mathring{W}$ is irreducible.*
5. *(5) $a = \partial\alpha$ and $b = \partial\beta$ do not bound discs in N.*
6. *(6) There is no essential disc in $N[a]$ or $N[b]$ which is contained in N.*

Then all of the following hold:

- *One of $N[a]$ and $N[b]$ is irreducible and is not a solid torus*
- *If one of $N[a]$ or $N[b]$ is reducible then the other is boundary-irreducible*
- *If $c_a \subset \partial M$ is a curve which compresses in $N[a]$ and if $c_b \subset \partial M$ is a curve which compresses in $N[b]$ then c_a and c_b cannot be isotoped in ∂M to be disjoint.*

Proof. We prove this theorem by contradiction. Without loss of generality, assume that $N[b]$ is reducible or boundary-reducible and that one of the following holds:

(a) $N[a]$ is reducible
(b) $N[a]$ is a solid torus
(c) $N[b]$ is reducible, $N[a]$ is irreducible, and a component of ∂M compresses in $N[a]$
(d) $N[a]$ is irreducible and there are disjoint curves c_a and c_b in ∂M which compress in $N[a]$ and $N[b]$ respectively.

(To see that this is the negation of the theorem, notice that if a torus component of $\partial N[a]$ compresses then either $N[a]$ is a solid torus or $N[a]$ contains an essential sphere.)

Let $\overline{R} \subset N[b]$ be an essential sphere or disc. If we assume (d) then choose $\overline{R}$ to be a disc with $\partial\overline{R} = c_b$, otherwise define $c_b = \partial\overline{R}$. Apply Theorem 3.2 to create a surface $\overline{Q}$. Out of all surfaces $\overline{Q}$ satisfying the conclusions of Theorem 3.2 choose $\overline{Q}$ so that $\widehat{q}(\overline{Q})$ is minimal. Theorem 3.2 guarantees that $\overline{Q}$ is still an essential sphere or disc. Notice that $\widehat{q} > 0$ by hypotheses (2) - (6). If $\overline{Q}$ is a disc with $\partial\overline{Q} \subset \partial_1 N[b]$, Theorem 3.2 guarantees that $\partial\overline{Q} = c_b$.

We need to place sutures on ∂M. We will use Lemma 3.1 and we start by defining curves c for use in that lemma. Let T be the union of torus components of ∂M. If $\overline{Q}$ is a sphere or disc with boundary on $\partial_0 N[b] \cup T$, define $c_\beta = \varnothing$. Otherwise, $\overline{Q}$ is a disc with boundary on $\partial_1 N[b] - T$. In which case, let $c_\beta = c_b$.

If no curve of ∂M disjoint from c_b compresses in $N[a]$, then let $c_\alpha = \varnothing$. If c_b compresses in $N[a]$, as well as in $N[b]$, let $c_\alpha = c_b$. If c_b does not compress in $N[a]$ but a curve c_a disjoint from c_b does, let $c_\alpha = c_a$. Notice that if c_α and c_β are distinct curves, then c_α and c_β are disjoint and non-parallel.

Define $c = c_\alpha \cup c_\beta$. We need to check that c satisfies the hypotheses of Lemma 3.1. By construction, $|c| \leq 2$. By the hypothesis that ∂M has at most one component which compresses in M, all components of c are on the same component of $\partial_1 N[b]$. If a component of c bounds an essential annulus with a curve of $\widehat{\gamma} \cup a$, then $L_\alpha \cup \overline{\alpha}$ intersects transversally an essential disc in M just once. This contradicts hypothesis (3).

Suppose that $|c| = 2$ and that c is the boundary of an essential annulus in N. Since $|c| = 2$, c_α and c_β are non-empty, disjoint, non-parallel curves. Since c_α compresses in $N[a]$ and since c_β is joined to c_α by an essential annulus, c_β also compresses in $N[b]$. But if $c_\beta = c_b$ compresses in $N[a]$ then c_α is defined to be c_b, contradicting the fact that $|c| = 2$. Thus, there is no essential annulus in N with boundary c.

Suppose that $|c| = 2$ and that there is an essential thrice-punctured sphere in N with boundary equal to $c \cup a$. Attaching α to a and a compressing disc to c_α creates a disc in $N[a]$ with boundary c_β. This contradicts the the fact that $|c| = 2$ since if c_β compresses in $N[a]$ then $c_\alpha = c_\beta$.

Thus c satisfies the criteria for an application of Lemma 3.1. Let $\gamma = \widetilde{\gamma} \cup \hat{\gamma}$ be the sutures on $\partial N[a]$ provided by that lemma. The sutured manifold

$(N, \gamma \cup a)$ is taut by Lemma 3.1 and hypotheses (4), (5), and (6). Since $\widehat{q}(\overline{Q}) > 0$, $\overline{Q}$ is not a sphere or disc disjoint from $\gamma \cup a$.

If assumption (a) holds, then $N[a]$ is reducible and so $(N[a], \gamma)$ is not taut. If assumptions (c) or (d) hold then, since c is disjoint from the sutures, $R_{\pm}(\gamma)$ compresses in $N[a]$ and so $(N[a], \gamma)$ is not taut. If assumption (b) holds then $M = S^3$ and L_α is the unknot. In this case, Lemma 5.3 guarantees that there is a taut conditioned surface in $(N, \gamma \cup a)$ which is disjoint from a and is a conditioned surface in $(N[a], \gamma)$. By hypothesis (6) and the assumption that L_α is the unknot, the surface is not taut in $(N[a], \gamma)$. Thus, we can apply Main Theorem B.

Conclusion (2) from Main Theorem B cannot hold because, by the construction of $\overline{Q}$, there is no compressing disc or a–boundary compressing disc for Q in N. Since $\overline{Q}$ is a sphere or disc, $-2\chi(\overline{Q}) < 0 \leq K(\overline{Q})$ by Lemma 5.2. Hence, conclusion (3) does not hold. Suppose, therefore that conclusion (1) holds.

$N[a]$ contains an essential separating sphere S that intersects $\overline{\alpha}$ twice and that cannot be isotoped to intersect $\overline{\alpha}$ fewer times. This sphere bounds a non-trivial homology ball in $N[a]$, and therefore in M. By hypothesis (2), S cannot be an essential sphere in M. Since N is irreducible, M does not have a spherical boundary component. Thus S bounds a ball B in M. Notice that this implies that M is a non-trivial homology sphere.

Since S is separating and B is not contained in $N[a]$, $\partial_0 N[a] \subset B$. Attaching $\eta(\overline{\alpha})$ to B produces a solid torus V containing W. Let $N_V = V - \mathring{\eta}(W)$ and notice that it is irreducible. Notice also that ∂V compresses in $N_V[a]$. Since V does not contain a non-trivial homology sphere connected summand, we may apply Main Theorem B (as in the previous paragraphs) to conclude that $N_V[b]$ is irreducible and that $\partial N_V[b]$ does not compress in $N_V[b]$. Thus, $\overline{Q}$ intersects ∂V. After an isotopy to eliminate inessential curves of intersection, an innermost disc of intersection D on $\overline{Q}$ is a compressing disc for ∂V contained outside V. If ∂D intersects a meridian curve of ∂V exactly once, ∂D runs exactly once along a regular neighborhood of $\overline{\alpha}$. D then guides an isotopy of $\overline{\alpha}$ into B, contradicting the construction of S. If ∂D is a meridional curve of ∂V, then W is contained in an $S^1 \times S^2$ summand of M. If ∂D intersects every meridional curve of ∂V more than once then W is contained in a lens space connected summand of M. By hypothesis, W intersects every reducing sphere in M, so M is $S^1 \times S^2$ or a lens space. Both possibilities contradict the fact that M is a homology sphere. This contradiction shows that conclusion (1) of Main Theorem B applied to $W \subset M$ cannot occur either. Hence, the theorem is proved. □

5.3. **Essential surfaces and boring knots and links.** For the remainder, we restrict to $M = S^3$ for simplicity. (This implies, for example that N is irreducible and that $\widetilde{\gamma} = \varnothing$.) For knots or links L_α and L_β which are related by boring, we desire to understand essential surfaces in the exterior of one by placing hypotheses on the other and on the embedding of W in S^3. In the proofs of the following theorems, the arguments showing that Main Theorem B can be applied are similar to those already given, so they will be abbreviated.

Theorem 5.5. *Suppose that L_α is a knot or link in S^3 obtained by boring a split link or unknot L_β using a handlebody W. Then either*

(1) L_α is a split link and $\overline{\alpha}$ intersects a splitting sphere once, or
(2) L_α is not a split link and there is a minimal genus Seifert surface for L_α with interior disjoint from $\overline{\alpha}$.

Proof. Suppose, first that α is separating and that $F - a$ is compressible in N. The boundary of a compressing disc is either parallel to a, in which case Conclusion (1) holds, or is a longitude of a component of $\partial_0 N[a]$. In this latter case, using the disc to compress that component of $\partial_0 N[a]$ produces a sphere intersecting $\overline{\alpha}$ once. Thus, (1) holds if α is separating and $F - a$ is compressible in N. Assume, therefore, that either α is non-separating or that $F - a$ is incompressible in N.

If α is separating, by assumption $F - a$ is incompressible in N. Recall that $\gamma = \widehat{\gamma}$ since $M = S^3$. If α is non-separating, $\gamma \neq \varnothing$. Since $M = S^3$ and since $F - (\gamma \cup a)$ consists of two thrice-punctured spheres each with meridional boundary, it is incompressible in N. Thus, in either case, by Lemma 3.1, $(N, \gamma \cup a)$ is taut.

Let $\overline{R}$ be an essential disc or sphere in $N[b]$ and let $\overline{Q}$ be the disc or sphere provided by Theorem 3.2. If $\widehat{q} = 0$ then, since N is irreducible, $\overline{Q}$ must be a disc. Discs are boundary-incompressible and so in no case is there a compressing disc or an a–boundary compressing disc for Q. $\overline{Q}$ cannot be disjoint from a, since either $\widehat{q} > 0$ or $\partial \overline{Q}$ intersects the meridional arcs of $a - b$. By Lemma 5.2, $K(\overline{Q}) \geq 0$. Hence, $-2\chi(\overline{Q}) < 0 \leq K(\overline{Q})$.

By Main Theorem B and Lemma 5.3, $N[a]$ is irreducible and L_α has a minimal genus Seifert surface disjoint from $\overline{\alpha}$ (that is, contained in N). □

This theorem can be immediately applied to understanding when split links and unknots can be obtained by boring split links and unknots.

Corollary 5.6. *Suppose that L_α is an split link or unknot obtained by boring a split link or unknot L_β using a handlebody $W \subset S^3$. Then the following are true:*

(1) *If L_α is a split link, then $\overline{\alpha}$ intersects a splitting sphere exactly once.*
(2) *If L_α is an unknot, then there is an unknotting disc for L_α with interior disjoint from $\overline{\alpha}$*
(3) *If L_β is a split link, then $\overline{\beta}$ intersects a splitting sphere exactly once.*
(4) *If L_β is an unknot, then there is an unknotting disc for L_β with interior disjoint from $\overline{\beta}$*

Proof. Theorem 5.5 gives the first two conclusions since an unknotting disc for L_α is a minimal genus Seifert surface if L_α is an unknot. Reversing the roles of α and β and applying Theorem 5.5 again produces the last two conclusions. This corollary also follows from Theorem 5.4. □

Corollary 5.7 ([ST2, Proposition 4.2])**.** *If $\overline{\alpha}$ is a tunnel for a tunnel number one knot or link L_α, L_α has a minimal genus Seifert surface disjoint from $\overline{\alpha}$.*

Proof. Every tunnel number one knot or link can be obtained by boring an unknot L_β using the standard unknotted genus two handlebody in S^3. A tunnel for a non-trivial tunnel number one knot or link is a boring arc for converting the knot or link into the unknot L_β. Thus, unless α is separating and $F - a$ is compressible in N, the corollary follows immediately from Theorem 5.5. In the remaining case, L_α is a split link and F is a genus two Heegaard surface for $N[a]$. The proof of the corollary in this situation is an easy exercise in Heegaard splitting theory. □

Remark. The proof of the previous corollary is not any better than Scharlemann and Thompson's proof. Indeed, their proof is certainly easier to understand than the arguments of this paper. However, it is interesting to note that they do rely on a theorem of Gabai which was proved using sutured manifold theory. The point of Theorem 5.5 is that a rather significant property of tunnel number one knots has a natural generalization to knots and links obtained by boring an unknot. Indeed, this theorem also generalizes the fact (due to Gabai and Scharlemann) that genus is superadditive under band sum (see Corollary 5.13).

Using Theorem 5.5, we can apply Main Theorem B again to obtain:

Theorem 5.8. *Suppose that L_β is a knot or link obtained by boring a split link or unknot L_α using a handlebody $W \subset S^3$. Assume that*

- *if L_α is a split link then $\overline{\alpha}$ does not intersect a splitting sphere just once.*
- *if L_α is an unknot then $\overline{\alpha}$ is not disjoint from the interior of an unknotting disc for L_α.*

then L_β is not a split link or unknot and L_β has a minimal genus Seifert surface $\overline{Q}$ with interior disjoint from $\overline{\beta}$ so that at least one of the following holds:

(1) there is an a–boundary compressing disc for $\overline{Q}$ in N.
(2) $-2\chi(\overline{Q}) \geq |\partial\overline{Q} \cap a| - |\partial\overline{Q} \cap \widehat{\gamma}|$

Remark. Corollary 5.12 rephrases this theorem for rational tangle replacements. Following that theorem, there is an example which shows that the possibility that there is an a–boundary compressing disc for $\overline{Q}$ cannot be eliminated. Notice that if $\overline{\beta}$ is isotopic to a non-trivial arc in $\overline{Q}$ then there is an a–boundary compressing disc for $\overline{Q}$ in S^3.

Proof. Suppose first that L_β is a split link or unknot. By Corollary 5.6, if L_α is a split link, $\overline{\alpha}$ intersects a splitting sphere exactly once and if L_α is an unknot, $\overline{\alpha}$ has interior disjoint from an unknotting disc for L_α. Both conclusions contradict our assumptions. Hence, L_β is not a split link or unknot.

By Theorem 5.5, applied with α and β reversed, there is a minimal genus Seifert surface $\overline{Q}$ for L_β which has interior disjoint from $\overline{\beta}$; that is, it is contained in N. Since $\overline{Q}$ is a Seifert surface for L_β, $\partial\overline{Q}$ lies on all components of $\partial N[b]$. This implies that $\partial\overline{Q}$ intersects the meridional arcs of $a - b$. Thus, $\overline{Q}$ is not disjoint from a.

Since $\overline{Q}$ is a minimal genus Seifert surface for L_β, it is incompressible. If there is an a–boundary compressing disc for $\overline{Q}$ in N, we have conclusion (1), so suppose that no such disc exists. If $-2\chi(\overline{Q}) < K(\overline{Q})$, Main Theorem B implies that $N[a]$ is irreducible. Also, in conjunction with Lemma 5.3, Main Theorem B implies that there is a minimal genus Seifert surface for L_α which is disjoint from $\overline{\alpha}$. Hence L_α is not a split link, and if L_α is an unknot then there is an unknotting disc disjoint from $\overline{\alpha}$. Hence, the assumption that $-2\chi(\overline{Q}) < K(\overline{Q})$ contradicts our hypotheses on L_α.

Thus, $-2\chi(\overline{Q}) \geq K(\overline{Q})$. Since $\overline{Q}$ is disjoint from $\overline{\beta}$, $q = q^* = 0$. The given inequality follows from the definition of $K(\overline{Q})$. □

With the stronger assumption that F is incompressible in N, we can restrict the possibilities for obtaining a non-hyperbolic knot or link from a split link by boring.

Theorem 5.9. *Suppose that L_β is a knot or link obtained by boring the link L_α using a handlebody $W \subset S^3$ with $N = S^3 - \mathring{W}$ boundary-irreducible. Suppose that L_α is a split link or that there is no minimal genus Seifert surface for L_α with interior disjoint from $\overline{\alpha}$. If the exterior of L_β contains an essential annulus or torus then one of the following holds:*

(1) $\Delta = 2$ and if there is an essential annulus then there is one which is either disjoint from or has meridional boundary on some component of L_β.
(2) There is an essential annulus in the exterior of L_β disjoint from $\overline{\beta}$ and which is either disjoint from or has meridional boundary on some component of L_β.
(3) There is an essential torus in N.

Example. It is possible to construct examples demonstrating that each of these conclusions can occur. An example of conclusion (1) is given by a (well-known) band sum of trivial knots which is a square knot. Figure 2 shows a split link L_α consisting of a trefoil (drawn so the "cabling" annulus is visible) and an unknot. There is an "S" shaped arc joining them. On the trefoil the annulus has boundary slope ± 6. Use the "S"-shaped arc to perform a handleslide of the unknot over the trefoil, following, but disjoint from, the boundary of the annulus. We now have a new link L_β where one component is the trefoil. By construction the cabling annulus for the trefoil persists into L_β. It is not difficult to show that the band move can be obtained by a boring operation with the exterior of the boring handlebody boundary-irreducible. In this example, $\Delta = 4$, the annulus in the exterior of L_β is disjoint from $\overline{\beta}$, and the annulus is disjoint from a component of L_β. This is conclusion (2) of the theorem. Finally, if N has an essential torus, then W must be in the solid torus V which it bounds in S^3. Applying Main Theorem B to $W \subset V$, produces conditions guaranteeing that ∂V is essential in $(V - \mathring{W})[b]$. The torus ∂V is then an essential torus in both N and $N[b]$. This is conclusion (3). See [T2] for more details.

Proof of Theorem 5.9. Corollary 5.6 shows that L_β is not an unknot or split link; consequently, there is no essential disc or sphere in $N[b]$. Let $\overline{R}$ be an essential annulus or torus in $N[b]$ and apply Theorem 3.2, obtaining a connected surface $\overline{Q}$. Since $\overline{Q}$ is not a sphere or disc and since $-\chi(\overline{Q}) \leq -\chi(\overline{R})$, $\overline{Q}$ is an annulus or torus. Since the genus of $\overline{Q}$ is no higher than the

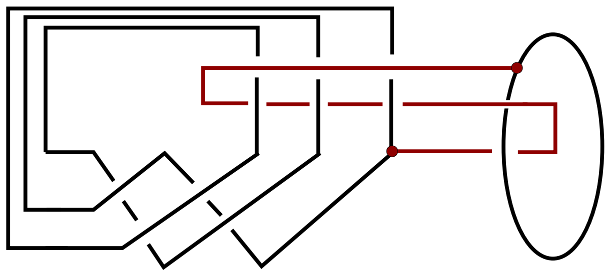

FIGURE 2. Performing a handleslide using the "S" shaped arc leaves the trefoil's essential annulus untouched.

genus of $\overline{R}$, if $\overline{R}$ was an annulus, then $\overline{Q}$ is an annulus. If $\overline{Q}$ is disjoint from $\overline{\beta}$ then it is contained in N. If it is a torus, we are done, so suppose that $\overline{Q}$ is an annulus disjoint from $\overline{\beta}$. In this case, if there is an a–boundary compression for $\overline{Q}$, N would contain an essential disc, contradicting the assumption that ∂W is incompressible in N.

We may, therefore, assume that there is no a–boundary compressing disc for the annulus or torus $\overline{Q}$. If $\overline{Q}$ is completely disjoint from a, then $\overline{Q}$ is disjoint from the meridional arcs of $a-b$. From our observations about meridional arcs, this means that $\overline{Q} \subset N$ and that either $\overline{Q}$ is a torus or $\overline{Q}$ is an annulus which is either disjoint from or has meridional boundary on one component of $N[b]$.

Suppose, therefore, that $\overline{Q}$ is not disjoint from a and that there is no compressing disc or a–boundary compressing disc for Q. By Main Theorem B, $-2\chi(\overline{Q}) \geq K(\overline{Q})$. Since $\chi(\overline{Q}) = 0$ and since $K(\overline{Q}) \geq 0$ (Lemma 5.2) we have $K(\overline{Q}) = 0$. Since α is separating in W, this implies

$$q(\Delta - 2) + q^*(\Delta^* - 2) + \Delta_\partial = 0.$$

Since each term is non-negative, each term must be zero. Hence $\Delta_\partial = 0$, implying that either $\overline{Q}$ is a torus or it is an annulus with boundary disjoint from or consisting of meridians on some component of $\partial S^3[\beta]$. If $q^* \neq 0$, then β is non-separating and we must have $\Delta^* = 2$. Since b^* intersects each meridional arc of $a-b$ at least twice, this means that there is exactly one such meridional arc. The number of meridional arcs is even, so this is a contradiction. If $q \neq 0$ then we have $\Delta = 2$. If both q and q^* are equal to zero, then since $\Delta_\partial = 0$, $\overline{Q}$ is an annulus disjoint from a, a possibility we have already considered. □

5.4. **Rational Tangle Replacements.** More results can be obtained by restricting attention to borings which are actually rational tangle replacements. Suppose that K is a knot or link in S^3 and $B' \subset S^3$ is a 3-ball which intersects K in two strands which form a rational tangle in B'. Let K' be a knot or link obtained by replacing $B' \cap K$ with a different rational tangle. We say that K' is obtained from K by rational tangle replacement (and *vice versa*). See [EM1] for definitions relating to rational tangles. It is easy to see that there are discs α and β in $W = \eta(K) \cup B'$ such that $L_\alpha = K'$ and $L_\beta = K$. Let $a = \partial\alpha$ and $b = \partial\beta$, as usual. The distance d between the two rational tangles is defined to be $\Delta(a,b)/2$.

Notice that rational tangle replacement is a type of boring if $d \geq 1$. We will generally use the language of boring when discussing rational tangle replacement. For details on how to convert statements phrased in terms of boring to statements phrased in terms of rational tangle replacement, see [T1, Lemma 7.2].

Define $\gamma \subset F$ with respect to a and b as before. In particular, if a is separating, then $\gamma = \varnothing$. Recall that if a is non-separating in F then $(N, \gamma \cup a)$ is taut and if a is separating and if $F - a$ is incompressible in N then (N, a) is taut.

The following lemmas are easy to prove; proofs can be found in [T1].

Tangle Calculations I (β separating)**.** *Suppose that L_β is a link obtained from L_α by a rational tangle replacement of distance d using W. Let $\overline{Q}$ be a suitably embedded surface in the exterior $N[b]$ of L_β. Let $\partial_1\overline{Q}$ be the components of $\partial\overline{Q}$ on one component of $N[b]$ and $\partial_2\overline{Q}$ be the components on the other. Let n_i be the minimum number of times a component of $\partial_i\overline{Q}$ intersects a meridian of $\partial N[b]$.*

- *If L_α is a link then*
$$K(\overline{Q}) \geq 2q(d-1) + d(|\partial_1\overline{Q}|n_1 + |\partial_2\overline{Q}|n_2).$$
- *If L_α is a knot then*
$$K(\overline{Q}) \geq 2q(d-1) + (d-1)(|\partial_1\overline{Q}|n_1 + |\partial_2\overline{Q}|n_2).$$

Tangle Calculations II (β non-separating)**.** *Suppose that L_β is a knot obtained from L_α by a rational tangle replacement of distance d using W. Let $\overline{Q}$ be a suitably embedded surface in the exterior $N[b]$ of L_β. Suppose that each component of $\partial\overline{Q}$ intersects a meridian of $N[b]$ n times.*

- *If L_α is a link then*
$$K(\overline{Q}) \geq 2q(d-1) + 2q^*(2d-1) + 2d|\partial\overline{Q}|n.$$

- *If L_β is a knot then*
$$K(\overline{Q}) \geq 2(d-1)(q+2q^*) + 2(d-1)|\partial\overline{Q}|n.$$

Our next observation concerns the implications of an a–boundary compressing disc. Although it doesn't appear in the statement, the lemma depends heavily on the fact that L_β is obtained from L_α by rational tangle replacement.

Lemma 5.10. *Suppose that $\overline{Q}$ is an essential surface in $N[b]$ disjoint from $\overline{\beta}$. Assume that $\partial\overline{Q}$ intersects the curve a minimally. If all components of $\partial\overline{Q}$ are meridians then there does not exist an a–boundary compressing disc joining two components of $\partial\overline{Q}$. If $\partial\overline{Q}$ has components on all components of $\partial N[b]$ and no component is a meridian, then if there is an a–boundary compression for $\overline{Q}$ in N the arc $\overline{\beta}$ is properly isotopic into $\overline{Q}$.*

Proof. Recall that since L_β is obtained from L_α by rational tangle replacement, all arcs of $a-b$ are meridional. Hence, if all components of $\partial\overline{Q}$ are meridians then $\partial\overline{Q}\cap a = \varnothing$. Suppose therefore that $\partial\overline{Q}$ intersects each component of $\partial N[b]$ and that no component of $\partial\overline{Q}$ is a meridian. Let D be an a–boundary compression. Let $\varepsilon = \partial D \cap \partial W$. It is a component of $a - \partial\overline{Q}$. Since $\overline{Q}$ is essential in $N[b]$, the arc runs at least once across b. Since no component of $\partial\overline{Q}$ is a meridian and since it intersects each component of $\partial N[b]$, each arc of $a-\partial\overline{Q}$ which runs across b does so exactly once. Hence, after pushing ε into W slightly, $\eta(\overline{\beta})$ can be viewed as a regular neighborhood of ε. Then D guides an isotopy of $\overline{\beta}$ into $\overline{Q}$. See Figure 3. □

The final preliminary lemma strengthens Theorem 3.2.

Lemma 5.11. *Suppose that L_β and L_α are related by a rational tangle replacement of distance $d \geq 1$ and that $\overline{R} \subset N[b]$ is an essential surface. Assume that $\partial\overline{R}$ is not disjoint from any component of $N[b]$ and that no component of $\partial\overline{R}$ is a meridian. Then $\overline{R}$ is isotopic to a properly embedded surface $\overline{Q}$ so that either $\overline{Q}$ is disjoint from $\overline{\beta}$ or $Q = \overline{Q}\cap N$ is a–boundary incompressible.*

Proof. The surface $\overline{Q}$ constructed by Theorem 3.2 is constructed by boundary compressing $R = \overline{R}\cap N$ using a–boundary compressing discs. Suppose that D is an a–boundary compressing disc for R. Let $\delta = \partial D \cap R$ and $\varepsilon = \partial D \cap \partial W$. Let $b_1, \ldots, b_q$ be the intersections of the interior of $\overline{R}$ with F. Each of these curves bounds a disc in W parallel to β.

Since all arcs of $a-b$ are meridional, and since ε can be isotoped to be a subarc of $a - \mathring{\eta}(b)$, ε does not join b_1 to b_q, unless $q = 2$ and $\varepsilon \subset \eta(b)$.

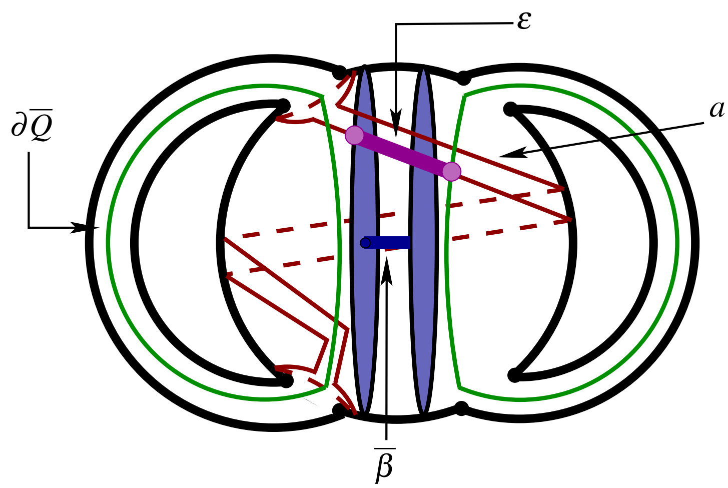

FIGURE 3. The arc $\overline{\beta}$ is parallel to ε.

If ε joins b_i to b_{i+1} then, as in the proof of [T1, Theorem 5.1] (which is the original source for Theorem 3.2), there is an isotopy of $\overline{R}$ which reduces $\widehat{q}$.

If ε joins $\partial\overline{R}$ to b_1 or b_q there is an isotopy of $\overline{R}$ which moves $\partial\overline{R}$ and which reduces q, as in [T1, Theorem 5.1]. This isotopy has the same effect as performing the boundary compression.

Thus, it remains to examine what happens when ε joins $\partial\overline{R}$ to itself or when it joins b_1 or b_q to itself. Since $\partial N[b]$ consists of tori and since $\overline{R}$ is essential, $\overline{R}$ is boundary-incompressible in $N[b]$. Thus, if ε joins $\partial\overline{R}$ to itself, the arc δ must be inessential in $\overline{R}$. There is, therefore, an isotopy of $\overline{R}$ which reduces q and accomplishes the boundary compression.

Suppose, therefore, that ε joins b_1 or b_q to itself. The arc ε is a subarc of a meridional arc of $a-b$. The arc ε is disjoint from $\partial\overline{R}$ and so is a meridional arc of $a-b$, and not just a subset of one. Let T be the component of $N[b]$ containing ε. Since $\varepsilon \cap \partial\overline{R} = \varnothing$, either $\partial\overline{R}$ is disjoint from T or $\partial\overline{R} \cap T$ consists of meridians. Thus, by hypothesis, this case cannot occur and $\overline{Q}$ is isotopic to $\overline{R}$. Since $q^* = 0$, $\overline{Q}$ is properly embedded and not just suitably embedded. □

Remark. It may be interesting to observe that the hypotheses on $\overline{R}$ in the statement of Lemma 5.11 can be weakened to the hypothesis that there is no essential annulus from $\overline{R}$ to a meridian of L_β. To see this, note that the only place the hypotheses on $\overline{R}$ are used is in the final paragraph of the proof. They are used to eliminate the possibility that there is an a–boundary compressing disc for R which joins b_1 or b_q to itself. If such an a–boundary compressing disc exists then it is possible to construct an essential annulus with one end on $\overline{R}$ and the other on a meridian of L_β.

The work of this paper can be used to give new proofs of several classical theorems about rational tangle replacements, including five out of the six theorems of [EM1]. This is done in [T2]. Here, we prefer to prove some new results; however, we do not shy away from showing how these results can give new proofs of some well-known theorems.

We begin by restating Theorem 5.8 for rational tangles:

Corollary 5.12. *Suppose that $L_\beta \subset S^3$ is a knot or link obtained by rational tangle replacement of distance $d \geq 1$ on a split link or unknot L_α. Assume that*

- *if L_α is a split link then $\overline{\alpha}$ does not intersect a splitting sphere just once*
- *if L_α is an unknot then $\overline{\alpha}$ is not disjoint from the interior of an unknotting disc for L_α,*

then L_β is not a split link or unknot and L_β has a minimal genus Seifert surface $\overline{Q}$ with interior disjoint from $\overline{\beta}$ so that at least one of the following holds:

(1) $\overline{\beta}$ is isotopic into $\overline{Q}$.
(2) $-\chi(\overline{Q}) \geq d$ and L_α is a split link
(3) $-\chi(\overline{Q}) \geq d-1$ and L_α is an unknot.

Proof. Apply Theorem 5.8 to produce the Seifert surface $\overline{Q}$ which is disjoint from $\overline{\beta}$. Isotope $\overline{Q}$ to minimize $|\partial\overline{Q} \cap a|$. By the proof of Theorem 5.8, either there is an a–boundary compressing disc for $\overline{Q} \subset N$ or $-2\chi(\overline{Q}) \geq K(\overline{Q})$. If the former happens, by Lemma 5.10, we conclude that $\overline{\beta}$ is properly isotopic into $\overline{Q}$. Suppose, therefore, that $-2\chi(\overline{Q}) \geq K(\overline{Q})$. Using the Tangle Calculations and the fact that $q = q^* = 0$ we see that if L_α is a link, then $-2\chi(\overline{Q}) \geq 2d$. If L_α is a knot, then $-2\chi(\overline{Q}) \geq 2(d-1)$. The given inequalities follow immediately. □

A pleasing corollary is Gabai and Scharlemann's result that genus is superadditive under band sum. A band sum is a rational tangle replacement of distance 1 on a split link.

Corollary 5.13 (Gabai [G3], Scharlemann [ScM4])**.** *Suppose that $K_1 \#_b K_2$ is the band sum of knots K_1 and K_2. Then*

$$\text{genus}(K_1 \#_b K_2) \geq \text{genus}(K_1) + \text{genus}(K_2)$$

with equality only if K_1 and K_2 have minimal genus Seifert surfaces disjoint from the band.

Proof. The statement holds if the band sum is a connected sum (i.e. if the band intersects a splitting sphere exactly once), so we may assume that the band intersects every essential sphere in the exterior of $L_\alpha = K_1 \cup K_2$ more than once. Let $W = \eta(K_1 \cup K_2 \cup b)$ where b is the band. (Note the ambiguity associated with the letter 'b' in this context.) Let α be a disc in $\eta(b)$ intersected once transversally by the core of b. Let β be a disc intersecting α once and which is "parallel" to the cocore of the band so that $L_\beta = K_1 \#_b K_2$. Since the band sum is not a connected sum, $F - a$ is incompressible in $S^3 - \mathring{W}$. Applying Corollary 5.12, we produce a minimal genus Seifert surface $\overline{Q}$ for L_β which is disjoint from $\overline{\beta}$, the cocore of the band. The proof now proceeds as in [G3] and [ScM4]. □

Superadditivity of genus under band sum provides a more interesting estimate of the genus of a knot L_β obtained by a rational tangle replacement on a split link L_α than does Corollary 5.12. To see this, suppose that δ is a disc in W which intersects α once, so that L_δ is obtained by a band sum of the components of L_α. To produce L_β, we can start with L_α, attach a band to create L_δ, and then insert into the band a 2–bridge knot or link K_3 corresponding to the tangle arising from the disc β. Let K_1 and K_2 be the components of L_α. By moving K_3 along the band so that it is close to K_2, we see that $L_\beta = K_1 \#_b (K_3 \# K_2)$. Thus, by supperadditivity of genus under band sum,

$$\text{genus}(L_\beta) \geq \text{genus}(K_1) + \text{genus}(K_3) + \text{genus}(K_2).$$

The result of Corollary 5.12 for L_α an unknot, likewise, can be interpreted as a result about attaching a band to a 2–bridge knot or link. However, not every such band attachment can be described as a rational tangle replacement on the unknot.

The superadditivity of genus under band sum is used in the next example to show that the possibility that $\overline{\beta}$ is isotopic into $\overline{Q}$ cannot be removed from Corollary 5.12.

Example. Figure 4 depicts the diagram of a 9_{37} knot L_β [K, CL]. The indicated rational tangle replacement converts L_β into a split link L_α. The rational tangle replacement has distance $d = 5$. In the diagram, it is not difficult to find a Seifert surface S for L_β consisting of an annulus and three twisted bands. Two of the bands have one half twist each and the third has three half twists. Thus, $-\chi(S) = 3$ and genus$(S) = 2$. L_β is the band sum of the unknot with a figure eight knot. The band is not disjoint from Seifert surfaces for the unknot and the figure eight knot. Hence, by Corollary 5.13, $\overline{Q}$ is a minimal genus Seifert surface for L_β. This is an example where L_α is a split link but conclusion (2) of Corollary 5.12 does not hold. It is easy to see that $\overline{\beta}$ is isotopic into $\overline{Q}$, which is conclusion (1) of Corollary 5.12.

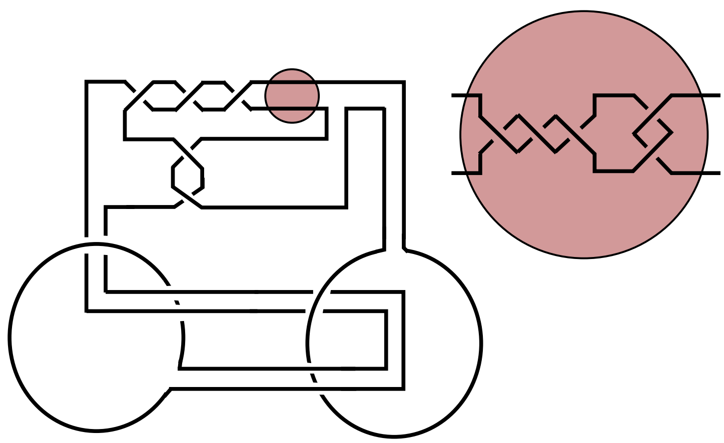

FIGURE 4. The knot L_β and a rational tangle replacement.

Remark. Scharlemann and Thompson [ST2] have shown that, in many cases, a tunnel for a tunnel number 1 knot can be isotoped and slid to lie in a minimal genus Seifert surface for the knot. Since tunnel number 1 knots are those knots which are obtained by boring the unknot or unlink using an unknotted handlebody, perhaps the first possible conclusion of Corollary 5.12 points to a more general phenomenon.

We now turn to an examination of genus zero and one surfaces in the exterior of knots and links obtained by a rational tangle replacement on a split link or unknot. The next theorem is a more sophisticated version of [T1, Theorem 7.3].

Theorem 5.14. *Suppose that L_β is a knot or link obtained by a rational tangle replacement of distance $d \geq 1$ on the knot or link L_α. Suppose that L_α is a split link or does not contain a minimal genus Seifert surface with interior disjoint from $\overline{\alpha}$. If L_α is a split link, also assume that $\overline{\alpha}$ does not intersect a splitting sphere just once.*

Suppose that $\overline{Q}$ is an essential properly embedded meridional planar surface in the exterior of L_β chosen so that out of all such surfaces $|\partial\overline{Q}|$ is minimal and, subject to that constraint, $|\overline{Q} \cap \overline{\beta}|$ is minimal. Then either $\overline{Q}$

is disjoint from $\overline{\beta}$ *or*

$$|\overline{Q} \cap \overline{\beta}|(d-1) \leq |\partial \overline{Q}| - 2$$

Proof. The proof is a fun exercise using the machinery developed so far. Alternatively, adapt the proof of [T1, Theorem 7.3] by replacing the appeal to main theorem of that paper with an appeal Main Theorem B of this paper. □

A crossing change or generalized crossing change of a knot K is achieved by choosing a disc $D \subset S^3$ which is pierced twice by K with opposite sign and by performing a $\pm 1/n$ Dehn surgery on ∂D with $n \in \mathbb{N}$. If $n = 1$, the new knot is obtained by changing the crossing of K. A generalized crossing change can be achieved by rational tangle replacement of distance $d = 2n$.

Corollary 5.15 (Scharlemann [ScM1], Scharlemann and Thompson [ST1])**.** *No generalized crossing change on a composite knot will produce the unknot.*

Proof. Suppose that L_β is a composite knot which can be converted to an unknot by a generalized crossing change. Let D be a crossing disc for L_β such that $\pm 1/n$ surgery on ∂D converts L_β to the unknot L_α. Let $W = \eta(L_\beta \cup D)$ and notice that L_α can be obtained from L_β by a rational tangle replacement of distance $d = 2n$. Notice that α is non-separating. Let $\overline{Q}$ be an essential meridional annulus in $N[b]$, chosen so that $|\overline{Q} \cap \overline{\beta}|$ is minimal.

Case 1: There is no unknotting disc for L_α with interior disjoint from $\overline{\alpha}$. The hypothesis of this case implies that L_α does not have a minimal genus Seifert surface with interior disjoint from $\overline{\alpha}$. Since α is non-separating in W, the hypotheses of Theorem 5.14 are satisfied. By Theorem 5.14, either $\overline{Q}$ is disjoint from $\overline{\beta}$ or

$$|\overline{Q} \cap \overline{\beta}|(d-1) \leq |\partial \overline{Q}| - 2 = 0.$$

Since the generalized crossing change is non-trivial, d is an even positive number. Thus, $\overline{Q}$ is disjoint from $\overline{\beta}$. But this implies that the generalized crossing change is taking place on one of the summands of L_β and so cannot unknot L_β. This contradiction shows that this case cannot occur.

Case 2: There is an unknotting disc for L_α with interior disjoint from $\overline{\alpha}$.

Suppose that $E \subset N$ is such a disc. Let (B, τ) be the tangle complementary to where the rational tangle replacement is taking place. That is, $B = S^3 - \mathring{\eta}(D)$ and $\tau = L_\beta \cap B$. E intersects ∂B in two arcs. Thus, either $E \cap B$

consists of two discs which can be used to isotope the strands of τ into ∂B or $E \cap B$ consists of a single disc which gives a parallelism between the strands of τ. In the former case, (B, τ) is a rational tangle and so L_β is a 2–bridge knot. By Schubert's theorem [ScH, ScJ], L_β cannot be composite.

Subject to the constraint that $\overline{Q}$ intersects $\overline{\beta}$ minimally, isotope $\overline{Q}$ so that $|Q \cap E|$ is minimal. Since $\partial\overline{Q} \cap \partial E \neq \varnothing$, Q is boundary-compressible by a disc E' bounded by an arc of $Q \cap E$ which is outermost on E.

Let $b_1, \ldots, b_q$ be the components of $\partial Q - \partial\overline{Q}$ labelled in order around $\eta(b) \subset F$. Let $\varepsilon = \partial E' \cap F$. If ε lies in an annulus component of $F - \operatorname{int}\overline{Q}$, then the disc E' can be used to isotope $\overline{Q}$ so that it intersects $\overline{\beta}$ fewer times. Similarly, if ε joins $\partial\overline{Q}$ to b_1 or b_q then there is an isotopy of $\overline{Q}$ moving $\partial\overline{Q}$ which reduces $|\overline{Q} \cap \overline{\beta}|$. Thus, ε must join $\partial\overline{Q}$ to itself. Since $\overline{Q}$ is an essential meridional annulus, the disc E' cannot be a boundary compressing disc for $\overline{Q}$ in $N[b]$, so compressing $\overline{Q}$ using E' creates two surfaces: an annulus and a disc with inessential boundary on $N[b]$. The annulus is isotopic to $\overline{Q}$ and intersects $\overline{\beta}$ fewer times than does $\overline{Q}$, contradicting the choice of $\overline{Q}$. Thus, this case cannot occur either. □

If a non-trivial surgery on a hyperbolic knot or link $L_\beta \subset S^3$ produces a manifold containing an essential sphere or torus, it is easy to show that the exterior of L_β contains an essential planar surface or punctured torus. The final theorem examines the possibilities for such surfaces in the exterior of a knot L_β obtained by rational tangle replacement on a split link or knot without a minimal genus Seifert surface disjoint from the boring arc.

Theorem 5.16. *Suppose that L_β is a knot or link obtained by rational tangle replacement of distance $d \geq 1$ on a knot or link L_α using handlebody W. Assume one of the following:*

- *L_α is a split link and $\overline{\alpha}$ does not intersect a splitting sphere just once*
- *L_α is a knot or link and there is not a minimal genus Seifert surface for L_α with interior disjoint from $\overline{\alpha}$.*

Suppose that L_β contains an essential planar surface or punctured torus $\overline{R}$ in its exterior. Then $\overline{R}$ can be properly isotoped so that one of the following is true:

(1) L_β is a link and $\partial\overline{R}$ is disjoint from some component of L_β.
(2) $\overline{R}$ has meridional boundary on some component of L_β.
(3) $\overline{R}$ is disjoint from $\overline{\beta}$ and $\overline{\beta}$ can be slid and isotoped to lie in $\overline{R}$.

(4) L_β and L_α are both links, $d = 2$, and $\overline{R}$ is a punctured torus disjoint from $\overline{\beta}$ with integer slope on both components of $\partial N[b]$.
(5) L_β is a link, L_α is a knot, $d \leq 2$, and $\overline{R}$ is a planar surface.
(6) L_β is a link, L_α is a knot, $d \leq 3$, and $\overline{R}$ is a punctured torus.
(7) L_β is a knot, L_α is a link, $d = 1$, $\overline{R}$ is a punctured torus with $\partial\overline{R}$ having integer slope.
(8) L_β and L_α are both knots, $d = 1$ and $\overline{R}$ is a planar surface.
(9) L_β and L_α are both knots, $d \leq 2$ and $\overline{R}$ is a punctured torus.

Proof. By Theorem 5.5 there is no essential sphere or disc in the exterior of L_β. Apply Theorem 3.2 to $\overline{R}$ obtaining a connected suitably embedded surface $\overline{Q}$. Suppose that neither conclusion (1) nor conclusion (2) holds. By Lemma 5.11, $\overline{Q}$ is isotopic to $\overline{R}$. In particular, it is also a planar surface or punctured torus and $\partial\overline{Q}$ has the same slope as $\partial\overline{R}$.

If $\overline{Q}$ is disjoint from $\overline{\beta}$ and if there is an a–boundary compressing disc for $\overline{Q}$ then, by Lemma 5.10, $\overline{\beta}$ is isotopic into $\overline{Q}$. This is conclusion (3). By the construction of $\overline{Q}$, we may assume that there is no a–boundary compressing disc for Q.

If $\overline{Q}$ is disjoint from a then it is disjoint from all meridional arcs of $a - b$ and so must have meridional boundary or must be disjoint from some component of $\partial N[b]$, contradicting our denial of (1) and (2). Hence, we may apply Main Theorem B to conclude that $-2\chi(\overline{Q}) \geq K(\overline{Q})$. Let $s = 2$ if $\overline{Q}$ is a planar surface and let $s = 0$ if $\overline{Q}$ is a punctured torus. We now consider the possibilites for α and β. We use the notation and results of the Tangle Calculation Lemmas.

Case 1: β and α are both separating. In this case, notice that $d \geq 2$. Since $-2\chi(\overline{Q}) = -2s + 2(|\partial_1\overline{Q}| + |\partial_2\overline{Q}|)$ we have

$$-2s + 2(|\partial_1\overline{Q}| + |\partial_2\overline{Q}|) \geq 2q(d-1) + d(|\partial_1\overline{Q}|n_1 + |\partial_2\overline{Q}|n_2).$$

Rearrange this to obtain

$$-2s \geq 2q(d-1) + |\partial_1\overline{Q}|(dn_1 - 2) + |\partial_2\overline{Q}|(dn_2 - 2).$$

If $\overline{Q}$ is a planar surface, then we must have either $dn_1 < 2$ or $dn_2 < 2$. Since d, n_1, and n_2 are all non-zero by hypothesis, we contradict the observation that $d \geq 2$. Hence $\overline{Q}$ is not a planar surface.

If $\overline{Q}$ is a punctured torus, then we must have $dn_1 \leq 2$ and $dn_2 \leq 2$. Since $d \geq 2$, we must have $d = 2$ and $n_1 = n_2 = 1$. This is conclusion (4).

Case 2: β is separating and α is non-separating. We have

$$-2s+2(|\partial_1\overline{Q}|+|\partial_2\overline{Q}|) \geq 2q(d-1)+(d-1)(|\partial_1\overline{Q}|n_1+|\partial_2\overline{Q}|n_2).$$

Rearranging, we obtain

$$-2s \geq 2q(d-1)+|\partial_1\overline{Q}|((d-1)n_1-2)+|\partial_2\overline{Q}|((d-1)n_2-2).$$

Therefore, we have $(d-1)n_1 \leq 2$ or $(d-1)n_2 \leq 2$. If $\overline{Q}$ is a planar surface the inequalities are strict. This produces conclusion (5). Otherwise, we obtain conclusion (6).

Case 3: β is non-separating and α is separating. Recall that $q^* = 0$ since $\overline{Q}$ is properly embedded. Now we have,

$$-2s+2|\partial\overline{Q}| \geq 2q(d-1)+2d|\partial\overline{Q}|n.$$

Rearranging we find

$$-2s \geq 2q(d-1)+|\partial\overline{Q}|(2dn-2).$$

Since d, n, and $|\partial\overline{Q}|$ are all positive, $s=0$ and $d=n=1$. This is conclusion (7).

Case 4: β and α are both non-separating. Finally, we have

$$-2s \geq 2q(d-1)+2|\partial\overline{Q}|((d-1)n-1)$$

If $s=2$, then $(d-1)n<1$ implying $d=1$. If $s=0$, then $(d-1)n \leq 1$. This implies $d \leq 2$. These are conclusions (8) and (9). □

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