

## Notes on Determinants

### 1. DETERMINANTS AND ROW OPERATIONS

**Theorem 1.** Suppose that an  $n \times n$  matrix  $A'$  is obtained from  $A$  by a single row operation. Then the following are true:

- (1) If the row operation switches two rows, then  $\det(A') = -\det(A)$ .
- (2) If the row operation scales a row by  $k \neq 0$ , then  $\det(A') = k \det(A)$ .
- (3) If the row operation replaces a row  $i$  by the sum of row  $i$  and row  $j$ , then  $\det(A') = \det(A)$ .

We prove only the third possibility. Our method of proof is by induction.

**Base Case:**  $n = 2$

Suppose that  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  and  $A' = \begin{bmatrix} a & b \\ a+c & b+d \end{bmatrix}$ . Then

$$\det(A) = ad - bc$$

and

$$\det(A') = a(b+d) - b(a+c) = ab + ad - ba - bc = ad - bc,$$

so the theorem holds. If  $A' = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix}$ , then  $\det(A') = d(a+c) - c(b+d) = ad - bc$ , as desired.

**Inductive Case:** Suppose that the theorem is true for  $n \times n$  matrices and that we want to prove it for  $(n+1) \times (n+1)$  matrices with  $n \geq 2$ .

Our row operation replaces row  $i$  with the sum of row  $i$  and row  $j$ . By an important theorem, we can calculate  $\det(A')$  by LaPlace expansion across row  $k$ , where  $k \neq i, j$  (using the fact that  $n+1 \geq 3$ ). The formula for LaPlace expansion tells us that

$$\det(A') = \sum_{c=1}^{n+1} a'_{kc} \det(A'_{kc})$$

where  $a'_{kc}$  is the entry in row  $k$  and column  $c$  of  $A'$ . Since row  $k$  of  $A'$  and row  $k$  of  $A$  are exactly the same  $a'_{kc} = a_{kc}$  for all  $c$ . The matrix  $A'_{kc}$  is an  $n \times n$  matrix which is obtained from the matrix  $A_{kc}$  by a row replacement operation. Since we are assuming the theorem is true for  $n \times n$  matrices,  $\det(A'_{kc}) = \det(A_{kc})$ . Hence,

$$\det(A') = \sum_{c=1}^{n+1} a'_{kc} \det(A'_{kc}) = \sum_{c=1}^{n+1} a_{kc} \det(A_{kc}) = \det(A).$$

□

This result has an important corollary:

**Corollary 2.** If  $A$  is an  $n \times n$  matrix and if  $E$  is an elementary matrix, then  $\det(EA) = \det(E) \det(A)$ .

*Proof.* Notice, first, that  $\det(I_n) = 1$ . Thus, if  $E$  is obtained from  $I_n$  by a row swap,  $\det(E) = -1$ . If  $E$  is obtained from  $I_n$  by scaling a row of  $I_n$  by  $k \neq 0$ ,  $\det(E) = k$ . And if  $E$  is obtained from  $I_n$  by a row replacement operation,  $\det(E) = 1$ . The matrix  $EA$  is obtained by performing the row operation corresponding to  $E$  on  $A$ , thus, if the row operation is a row swap,  $\det(EA) = -\det(A)$ ; if the row operation scales a row of  $A$  by  $k \neq 0$ , then  $\det(EA) = k \det(A)$ ; and if the row operation is a row replacement operation,  $\det(EA) = \det(A)$ . The result follows immediately.  $\square$

We can now prove the most important theorem about determinants:

**Theorem 3.** *If  $A$  and  $B$  are  $n \times n$  matrices,  $\det(AB) = \det(A) \det(B)$ .*

*Proof.* Let  $E_1, \dots, E_p$  be the elementary matrices so that  $E_p E_{p-1} \dots E_1 A = \text{rref}(A)$ .

**Case 1:**  $A$  is invertible.

In this case  $\text{rref}(A) = I_n$  and  $E_p \dots E_1 A = I_n$ . That is,  $A = E_1^{-1} \dots E_p^{-1}$ . Then taking the determinant of  $AB = E_1^{-1} \dots E_p^{-1} B$  gives us:

$$\det(AB) = \det(E_1^{-1} \dots E_p^{-1} B).$$

By Corollary 2, since the inverse of an elementary matrix is an elementary matrix:

$$\det(AB) = \det(E_1^{-1}) \dots \det(E_p^{-1}) \det(B).$$

Applying Corollary 2 again,

$$\det(AB) = \det(E_1^{-1} \dots E_p^{-1}) \det(B).$$

And this is just,

$$\det(AB) = \det(A) \det(B)$$

as desired.

**Case 2:**  $A$  is not invertible.

In this case,  $\text{rref}(A)$  is not  $I_n$  and so must have a row of all zeroes. Performing Laplace expansion across this row shows that  $\det(\text{rref}(A)) = 0$ . We have, (using Corollary 2),

$$(*) \quad 0 = \det(\text{rref}(A)) = \det(E_p \dots E_1 A) = \det(E_p) \dots \det(E_1) \det(A).$$

If  $E$  is an elementary matrix, then  $E$  is invertible. Hence, by Corollary 2

$$1 = \det(I_n) = \det(E) \det(E^{-1}).$$

This implies that the determinant of an elementary matrix is not zero. Hence, Equation (\*) shows us that  $\det(A) = 0$ . In other words, the determinant of a non-invertible matrix is zero.

If  $A$  is not invertible, neither is  $AB$  by Exercise 34 of Section 2.4. Hence,  $\det(AB) = 0$ . Thus,  $\det(AB) = \det(A) \det(B) = 0$ .  $\square$