

MA 122: Exam 2 Solutions

Problem 1: Let $f(x) = \cos(x)$. Recall that the $2n$ th Maclaurin Polynomial for $f(x)$ is given by

$$P_{2n}(x) = \sum_{i=0}^n \frac{(-1)^i x^{2i}}{(2i)!}.$$

Let $P(x)$ be the series

$$P(x) = \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i}}{(2i)!}.$$

- (a) (10 pts) Analyze the error terms of the Maclaurin polynomials to show that $P(x) = f(x)$ for all $x \in \mathbb{R}$.

Solution: Recall that $f^{(n+1)}(t)$ is one of $\pm \cos(t)$ or $\pm \sin(t)$. Thus, for t between 0 and x we have

$$|f^{(n+1)}(t)| \leq 1.$$

Consequently, by Taylor's theorem:

$$|P_n(x) - f(x)| \leq \frac{1}{(n+1)!} |x|^{n+1}$$

for all n and x . We know that

$$\lim_{n \rightarrow \infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

for all x . Thus, by the squeeze theorem,

$$\lim_{n \rightarrow \infty} |P_n(x) - f(x)| = 0.$$

That is, $\lim_{n \rightarrow \infty} P_n(x) = f(x)$. But by definition, $\lim_{n \rightarrow \infty} P_n(x) = P(x)$. Thus, $P(x) = f(x)$ for all x .

(b) (10 pts) **Find** a series which represents the function

$$g(x) = \cos(x^3)$$

for $x \in \mathbb{R}$. **Explain** why this series converges to $g(x)$ for all $x \in \mathbb{R}$.

Solution: From (a) we know that

$$P(x) = f(x)$$

for all $x \in \mathbb{R}$. This means that for all $x \in \mathbb{R}$:

$$P(x^3) = f(x^3)$$

which is the same as saying

$$\sum_{i=0}^{\infty} \frac{(-1)^i x^{6i}}{(2i)!} = \cos(x^3)$$

for all x .

(c) (10 pts) Find a series which represents the function

$$h(x) = \int_0^x t \cos(t^3) dt$$

for $x \in \mathbb{R}$. **You do not need to explain your work.**

Solution: We know that

$$\cos(t^3) = \sum_{i=0}^{\infty} \frac{(-1)^i t^{6i}}{(2i)!}$$

Thus,

$$t \cos(t^3) = t \sum_{i=0}^{\infty} \frac{(-1)^i t^{6i}}{(2i)!} = \sum_{i=0}^{\infty} \frac{(-1)^i t^{6i+1}}{(2i)!}.$$

Now we integrate,

$$\begin{aligned} \int_0^x t \cos(t^3) dt &= \int_0^x \sum_{i=0}^{\infty} \frac{(-1)^i t^{6i+1}}{(2i)!} dt \\ &= \sum_{i=0}^{\infty} \int_0^x \frac{(-1)^i t^{6i+1}}{(2i)!} dt \\ &= \sum_{i=0}^{\infty} \frac{(-1)^i x^{(6i+2)}}{(2i)!(6i+2)} \end{aligned}$$

Problem 2: Let $f(x, y) = \frac{x}{y}$.

- (a) (6 pts) Graph three y -slices of $f(x, y)$ using y -values of -2 , $-1/2$, and 1 .

- (b) (4 pts) Use your answer from (a) to explain the meaning of

$$f_x(4, -2) = -1/2.$$

Solution: $f_x(4, -2)$ is the slope of the line tangent at $x = 4$ to the slice of the graph of $f(x, y)$ in the $y = -2$ slice. Since the slice is a line, this slope is just the slope of the line: $-1/2$.

Problem 3:

- (a) (10 pts) Find the equation of the plane which contains the point $(1,2,3)$ and is perpendicular to the vector $(-2, 1, 5)$. **Explain** why this is the correct formula for the plane.

Solution: The plane containing the origin which is perpendicular to the vector $(-2, 1, 5)$ is the set of all points (x, y, z) such that $(x, y, z) \cdot (-2, 1, 5) = 0$. This is because the dot product of two vectors is zero if and only if the vectors are perpendicular. To obtain the plane perpendicular to $(-2, 1, 5)$ and which contains $(1, 2, 3)$, we simply translate our solution to

$$(x - 1, y - 2, z - 3) \cdot (-2, 1, 5) = 0.$$

- (b) (10 pts) Find the equation of the plane which contains the points $(0, 2, 3)$, $(1, 4, -6)$ and $(2, 3, -1)$.

Solution: Translate the given points by $-(0, 2, 3)$ to obtain the points $(0, 0, 0)$, $(1, 2, -9)$, and $(2, 1, -4)$. A normal vector to the plane containing these three points may be found by taking the cross product

$$(1, 2, -9) \times (2, 1, -4) = (1, -14, -3).$$

Thus the equation of the plane we are asked for is

$$(1, -14, -3) \cdot (x, y - 2, z - 3) = 0.$$

Problem 4: (10 pts) Atlas, Menoetius, and Prometheus are fighting over the earth. Atlas is pulling with a force of 100 N in the direction of the unit vector $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. Menoetius is pulling with a force of 150 N in the direction of the unit vector $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{6}})$. Prometheus is pulling in a direction and with a force such that the earth isn't moving. **Find** a vector representing the direction and magnitude of the force that Prometheus is applying to the earth.

Solution: The force used by Atlas can be represented by the vector

$$\mathbf{A} = 100(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}) = (100/\sqrt{3}, 100/\sqrt{3}, 100/\sqrt{3})$$

since Atlas uses a force with magnitude 100 N in the direction $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$.

Similarly, the force used by Menoetius can be represented by the vector

$$\mathbf{M} = (-150/\sqrt{3}, 150/\sqrt{2}, -150/\sqrt{6}).$$

Let \mathbf{P} represent the force used by Prometheus, since the earth is unmoved we have:

$$\begin{aligned} \mathbf{P} &= -(\mathbf{A} + \mathbf{M}) \\ &= -\left(\frac{100}{\sqrt{3}} - \frac{150}{\sqrt{3}}, \frac{100}{\sqrt{3}} + \frac{150}{\sqrt{2}}, \frac{100}{\sqrt{3}} - \frac{150}{\sqrt{6}}\right). \end{aligned}$$

Problem 5: Let

$$f(x, y) = \begin{cases} x & \text{if } y = 0 \\ y & \text{if } x = 0 \\ 2(x + y) & \text{if neither of } x, y \text{ equals } 0 \end{cases}$$

For example,

$$\begin{aligned} f(2, 0) &= 2 \\ f(0, -2) &= -2 \\ f(1, 1) &= 4 \end{aligned}$$

It is a fact that $f_x(0, 0) = 1$ and $f_y(0, 0) = 1$.

(a) (10 pts) Let $L(x, y) = x + y$. Show that

$$\lim_{(x,y) \rightarrow (0,0)} \frac{f(x, y) - L(x, y)}{\|(x, y) - (0, 0)\|} \neq 0$$

Solution: It suffices to show that

$$\lim_{(x,x) \rightarrow (0,0)} \frac{f(x, y) - L(x, y)}{\|(x, y) - (0, 0)\|} \neq 0.$$

For $x \neq 0$, we have $f(x, x) = 4x$ and $L(x, x) = 2x$. Thus, when $x \neq 0$,

$$\frac{f(x, x) - L(x, x)}{\|(x, x)\|} = \frac{2x}{\sqrt{2x^2}} = \frac{2x}{|x|\sqrt{2}}.$$

Thus,

$$\lim_{(x,x) \rightarrow (0,0)} \frac{f(x, y) - L(x, y)}{\|(x, y) - (0, 0)\|} = \lim_{x \rightarrow 0} \frac{2x}{|x|\sqrt{2}} = \pm \frac{2}{\sqrt{2}} \neq 0$$

(b) (10 pts) Explain why (a) shows that f is not differentiable. (Your answer should involve the concept of “relative error”.)

Solution: By definition, f is differentiable at $(0, 0)$ if and only if both partial derivatives of f exist at $(0, 0)$ (which they do) and if the limit of relative error between f and L is 0. In (a), we showed precisely that the limit of the relative error is not zero and so f is not differentiable at $(0, 0)$.

- (c) (10 pts) Let $\mathbf{v} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Use the limit definition of directional derivative to calculate $f_{\mathbf{v}}(0, 0)$.

Solution:

$$\begin{aligned} f_{\mathbf{v}}(0, 0) &= \lim_{h \rightarrow 0} \frac{f(\mathbf{0} + h\mathbf{v}) - f(\mathbf{0})}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(h/\sqrt{2}, h/\sqrt{2}) - 0}{h} \\ &= \lim_{h \rightarrow 0} \frac{4h/\sqrt{2}}{h} \\ &= \lim_{h \rightarrow 0} 4/\sqrt{2} \\ &= 4/\sqrt{2} \end{aligned}$$

Problem 6: Let $T(x, y) = \frac{60(x^2 + y^2)}{1 + x^2 + y^2}$ denote the temperature of a point (x, y) on a sheet of metal. The following calculations have been done for you:

$$\begin{aligned} T_x(x, y) &= \frac{120x}{(1+x^2+y^2)^2} & T_x(1, -1) &= \frac{40}{3} \\ T_y(x, y) &= \frac{120y}{(1+x^2+y^2)^2} & T_y(1, -1) &= -\frac{40}{3} \end{aligned}$$

- (a) (5 pts) At the point $(1, -1)$, we have $T_x(1, -1) \geq 0$. Explain what this means with regard to the temperature of the sheet of metal.

Solution: If we increase x slightly from 1 the temperature of the metal increases from $T(1, -1) = 40$.

- (b) (5 pts) At the point $(1, -1)$, we have $T_y(1, -1) \leq 0$. Explain what this means with regard to the temperature of the sheet of metal.

Solution: If we increase y slightly from 1 the temperature of the metal decreases from $T(1, -1) = 40$.

- (c) (5 pts) Explain why $T(x, y)$ is differentiable at all values of x and y .

Solution: We have a theorem which says that if both partial derivatives of T exist and are continuous everywhere, then the function is differentiable everywhere. It is evident from the formulae for T_x and T_y that they are continuous at every (x, y) point.

- (d) (5 pts) Find the equation of the tangent plane to the graph of $T(x, y)$ at the point $(1, -1, 40)$.

Solution:

$$z = (40/3, -40/3) \cdot (x - 1, y + 1) + 40.$$

- (e) (5 pts) In what direction from $(1, -1)$ is the temperature of the plate increasing the fastest?

Solution: In the direction given by the gradient

$$\nabla T(1, -1) = (40/3, -40/3).$$

Problem 7: (10 pts Extra-Credit) Give a thorough (but not necessarily rigorous) proof of the following:

Suppose that $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is differentiable at \mathbf{a} and that \mathbf{v} is a unit vector. Then

$$f_{\mathbf{v}}(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{v}.$$

Solution: See the practice exam for a solution.