Simplices

Definition 1. Suppose that $v_0, \dots, v_k \in \mathbb{R}^n$. The **convex hull** of $\{v_0, \dots, v_k\}$ is the smallest convex set containing $v_0, \dots, v_k$. It is denoted $CH(v_0, \dots, v_k)$. It turns out that

$$CH(v_0, \dots, v_k) = \left\{ w \in \mathbb{R}^n : \exists \lambda_0, \dots, \lambda_k \in \mathbb{R} \text{ s.t. } w = \sum \lambda_i v_i \text{ and } \sum \lambda_i = 1 \right\}.$$

Definition 2. A **vector subspace** of $\mathbb{R}^n$ is a subset which is closed under (finite) linear combinations. An **affine subspace** is a subset which is the translation of a vector subspace. That is, W is an affine subspace of $\mathbb{R}^n$ if and only if there exists a vector subspace V and a vector a such that for each $w \in W$ there exists $v \in V$ so that $w = a + v$. We denote this

$$W = a + V.$$

The dimension of W is defined to be the dimension of V (as a subspace of $\mathbb{R}^n$ – this is not topological dimension).

Definition 3. Suppose that $v_0, \dots, v_k \in \mathbb{R}^n$. Then $v_0, \dots, v_k$ are **affinely independent** iff for each collection of $m+1$ distinct points $w_0, \dots, w_m \in \{v_0, \dots, v_k\}$ there is no $m-1$ dimensional affine subspace containing $w_0, \dots, w_m$. Equivalently, $v_0, \dots, v_k$ are affinely independent if the vectors $v_1 - v_0, v_2 - v_0, \dots, v_k - v_0$ are linearly independent in $\mathbb{R}^n$.

Example. Two points are affinely independent if and only if they are not the same. Three points are affinely independent if and only if they are not collinear. $\mathbb{R}^n$ contains at most $n+1$ affinely independent points.

Definition 4. A k**–dimensional simplex** Δ in $\mathbb{R}^n$ is the convex hull of $k+1$ affinely independent points. If we need to specify the points, we will sometimes write $\Delta = \Delta(v_0, \dots, v_k)$, where $v_0, \dots, v_k$ are the affinely independent points. An l-dimensional **face** of Δ is the convex hull of distinct points $w_0, \dots, w_l \in \{v_0, \dots, v_k\}$. We consider $\varnothing$ to be a (-1)–dimensional face of every simplex. The **standard** n–simplex is the convex hull of $0, e_1, \dots, e_n$ where e_i is the ith standard basis vector of $\mathbb{R}^n$.

Definition 5. Suppose that $P \subset \mathbb{R}^n$ is the union of finitely many simplices $\mathscr{T}$ (not necessarily of the same dimension). Then $\mathscr{T}$ is a (geometric) **triangulation** of P if whenever σ, τ are simplices in $\mathscr{T}$ then $\sigma \cap \tau$ is a face of each.