HW due May 1

(1) Read pp 109-110 (skip the business about suspensions), 139 -144 in text

(2) Give a cut-and-paste proof that the projective plane $\mathbb{R}P^2$ is homeomorphic to the result of gluing a disc to a Möbius band. It should be more detailed than the one in the book.

(3) Let $\texttt{Homeo}(X)$ denote the group of all homeomorphisms of a topological space to itself. (The operation is composition of maps.) Let G denote all those homeomorphisms of which are homotopic to the identity map. Prove that G is a normal subgroup of $\texttt{Homeo}(X)$. This means that you should prove that it is a subgroup and that it is normal. (For the definition of normal subgroup, see the point-set topology notes.) The quotient group $\texttt{Homeo}(X)/G$ is called the mapping class group of X. The mapping class group of a surface is the subject of much current research in topology, geometry, and analysis.

(4) Let X be a path-connected topological space and let $p \in X$. Recall the definition of $\pi_1(X, p)$ from the previous homework. Suppose that $[f], [g] \in \pi_1(X, p)$. Define $[f] \cdot [g]$ as follows. First, define $f \cdot g \colon I \to X$ by

$$f \cdot g(s) = \begin{cases} f(2s) & 0 \leq s \leq 1/2 \\ g(2s-1) & 1/2 \leq s \leq 1. \end{cases}$$

Notice that $f \cdot g$ is a loop based at p. Define

$$[f] \cdot [g] = [f \cdot g]$$

(a) Prove that if f' is basepoint homotopic to f and if g' is basepoint homotopic to g, then $f' \cdot g'$ is basepoint homotopic to $f \cdot g$. This shows that $\cdot$ is a well defined operation on $\pi_1(X, p)$.

(b) Let $c \colon I \to X$ be the constant map $c(s) = p$ for all $s \in I$. Prove that $[c]$ is an identity element in $\pi_1(X, p)$. (This requires you to show, that for all based loops $f \colon I \to X$, $f \cdot c$ is basepoint homotopic to f.)

(c) Let $f \colon I \to X$ be a based loop. Define $\overline{f} \colon I \to X$ by $\overline{f}(s) = f(1-s)$. Prove that $[f] \cdot [\overline{f}] = [c]$ and $[\overline{f}] \cdot [f] = [c]$. This shows that $[f]$ and $[\overline{f}]$ are inverses in $\pi(X, p)$.

(d) Let $f, g, h \colon I \to X$ be based loops. Prove that

$$([f] \cdot [g]) \cdot [h] = [f] \cdot ([g] \cdot [h]).$$

(5) This problem will be useful in the next problem. Let J be the interval $[0, 1]$ thought of as a simplicial complex with vertices at 0, $1/2$,

and 1. Let J' be the simplex $[0,1]$. Suppose that $f\colon J \to J'$ is a simplicial map such that $f(0) = 0$. Prove the following:

(a) If $f(1/2) = f(0) = 0$ and $f(1) = 1$ then f is homotopic to the identity map by a homotopy $F_t(x)$ such that $F_t(0) = 0$ and $F_t(1) = 1$ for all t.

(b) If $f(1/2) = f(1) = 1$ then f is homotopic to the identity map by a homotopy $F_t(x)$ such that $F_t(0) = 0$ and $F_t(1) = 1$ for all t.

(c) If $f(1/2) = 1$ and $f(1) = f(0) = 0$ then f is homotopic to the constant map $x \to 0$ for all $x \in J$ by a homotopy $F_t(x)$ such that $F_t(0) = F_t(1) = 0$ for all t.

(6) This problem will be useful for calculation $\pi_1(S^1, p)$. Let S^1 be the square in $\mathbb{R}^2$ having vertices at $p = (1,0)$, $(0,1)$, $(-1,0)$, and $(0,-1)$. Suppose that $f\colon I \to S^1$ is a continuous function such that $f(0) = f(1) = p$.

(a) Explain why there is a triangulation $\mathcal{T}$ of I so that f can be homotoped to a simplicial map $g\colon I \to S^1$. Furthermore, the homotopy $F_t(x)$ can be chosen so that $F_t(0) = F_t(1) = p$ for all t. Let n denote the number of vertices of $\mathcal{T}$. Notice that $n = 2$ if and only if the only 1-simplex in $\mathcal{T}$ is I itself.

(b) Say that g **stalls** if $n \geq 3$ and if there exist adjacent vertices v_0, v_1 in I so that $g(v_0) = g(v_1)$. Prove that if g stalls, then there is a triangulation $\mathcal{T}'$ of I so that $\mathcal{T}'$ has fewer vertices than $\mathcal{T}$ and g can be homotoped to a map $g'\colon I \to S^1$ by a basepoint preserving homotopy such that g' is simplicial with respect to the given triangulation on S^1 and the triangulation $\mathcal{T}'$ of I.

(c) Say that g **backtracks** if $n \geq 3$ and if there exist adjacent vertices v_0, v_1, and v_2 in I so that $g(v_0) = g(v_2)$ and $g(v_0) \neq g(v_1)$. Prove that if g stalls, then there is a triangulation $\mathcal{T}'$ of I so that $\mathcal{T}'$ has fewer vertices than $\mathcal{T}$ and g can be homotoped to a map $g'\colon I \to S^1$ by a basepoint preserving homotopy such that g' is simplicial with respect to the given triangulation on S^1 and the triangulation $\mathcal{T}'$ of I.

(d) Use the previous parts of the problem to prove that f can be homotoped to a map $h\colon I \to X$ by a basepoint preserving homotopy so that either h is a constant map or I winds monotonically around S^1 with no stalling and no backtracking. In particular, for any two points y_1, y_2 in S^1, the sets $h^{-1}(y_1)$ and $h^{-1}(y_2)$ have the same number of points.