HW due May 8

For this homework, you will need the following concepts:

Suppose that G and G' are groups and that $\phi\colon G \to G'$ is a function. Then ϕ is a **homomorphism** if for all $a, b \in G$, $\phi(ab) = \phi(a)\phi(b)$. (Writing group elements next to each other denotes "do the group operation".) If ϕ is a bijective homomorphism it is called an **isomorphism**.

The notation $\mathbb{Z}^n$ denotes the group $\mathbb{Z} \times \mathbb{Z} \ldots \times \mathbb{Z}$ with component-wise addition. For example in $\mathbb{Z}^2$: $(1,2) + (-3,4) = (-2,6)$. You may use the fact that $\pi_1(S^1, p)$ is isomorphic to $\mathbb{Z}$.

If $\alpha\colon I \to X$ is a path let $\overline{\alpha}\colon I \to X$ be defined by $\overline{\alpha}(s) = \alpha(1-s)$. If $\alpha, \beta\colon I \to X$ are paths so that $\alpha(1) = \beta(0)$, define

$$\alpha \cdot \beta(s) = \begin{cases} \alpha(2s) & 0 \leq s \leq 1/2 \\ \beta(2s-1) & 1/2 \leq s \leq 1. \end{cases}$$

(1) Read pages 32-33, 139 - 149 of the text.
(2) Read pages 34-35, study figures 25 - 28 and read Theorem 1.18.
(3) Suppose that S is a surface and that $f\colon I \to S$ is a loop based at $p \in S$ such that the image of f is a simple closed curve in S. It is a fact that f is base-point homotopic to a constant map $I \to p$ if and only if the image of f is the boundary of a (topological) disc in S. Use this fact to find a based loop $f\colon I \to P^2$ so that f is not homotopic to a constant map but $f \cdot f$ is. This shows that $\pi_1(P^2, p)$ has an element of order 2.
(4) (Bonus) Suppose that X is a path connected topological space and that $p, q \in X$ are points in X. In this exercise you will prove that $\pi_1(X, p)$ is isomorphic to $\pi_1(X, q)$. Let $f\colon I \to X$ be a loop based at p and let $[f]$ be its class in $\pi_1(X, p)$. Let $\alpha\colon I \to X$ be a path from p to q. Using α, we will define a homomorphism $\phi\colon \pi_1(X, p) \to \pi_1(X, q)$. Define $\phi([f]) = [\overline{\alpha} \cdot f \cdot \alpha]$.
   (a) Think about the map ϕ and give an informal explanation for why it is a homomorphism.
   (b) Give an informal explanation for why ϕ is a surjection.
   (c) To show that ϕ is an injection, it suffices to show that ϕ has an inverse function. What is the inverse function?
(5) Let X and Y be path-connected topological spaces. Suppose that $f\colon X \to Y$ is a continuous function such that for a basepoint $x_0 \in X$, $y_0 = f(x_0)$. Consider the function

$$f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

defined by

$$f_*([\alpha]) = [f \circ \alpha]$$

(the $\circ$ means function composition). You may assume (or prove, if you wish) that f_* is well-defined.

(a) Prove that f_* is a homomorphism.

(b) Let $k \in \mathbb{N}$. Consider the map $f\colon [0,1]/\{0,1\} \to S^1 = Y$ given by $f(s) = (\cos(k2\pi s), \sin(k2\pi s))$. Recall that $[0,1]/\{0,1\}$ is homeomorphic to S^1. Thus both domain and range have π_1 isomorphic to $\mathbb{Z}$. Describe the subgroup of $\pi_1(Y)$ which is the image of f_*.

(6) Consider closed surfaces S and Σ such that there is a covering map $S \to \Sigma$ such that the preimage of each point in Σ consists of k points in S. Prove that $\chi(S) = k\chi(\Sigma)$. (Hint: If $\mathcal{T}$ is a triangulation of Σ, explain why the inverse image of $\mathcal{T}$ is a triangulation of S.)

(7) Use the previous problem to prove that the only closed orientable surface which can be covered by the torus T^2 is the torus. Also prove that if $S \to S$ is a covering map of a closed surface to itself then either the map is a homeomorphism or $S = T^2$ or $S = K^2$.