Problem Set 4 Answer Key

Problem A

In class we defined a function F which determines whether or not a certain configuration of the 15-puzzle is solvable. We proved that if F has a value of 1 then the configuration is not solvable. Give a careful, thorough, and complete explanation of why that is the case. In other words, explain our proof in your own words

Answer: We begin by defining F.

Suppose that g is the symmetry in $\mathbb{S}_{16}$ which produces the given configuration from the solved configuration. Let $\varepsilon(g) = +1$ if g is an even permutation (i.e. can be written as an even number of transpositions) and -1 if g is an odd permutation (i.e. can be written as an odd number of transpositions.)

Let $x(g)$ be the number of spaces that the blank square has moved to the left from the lower right corner and let $y(g)$ be the number of spaces it has moved up. Let $\delta(g) = +1$ if $x(g)+y(g)$ is even and let $\delta(g) = -1$ if $x(g)+y(g)$ is odd. Define $F(g) = \varepsilon(g)\delta(g)$.

Now we show that F is unchanged after a legal move.

A legal move interchanges the blank square and a square above, below, to the left, or to the right of it. It can be thought of as a transposition in $\mathbb{S}_{16}$ and so after a legal move, ε has changed sign. Similarly, either x has increased or decreased by 1 or y has increased or decreased by 1. Thus $x+y$ has changed from odd to even or vice versa. In other words, δ has also changed sign. Since F is the product of ε and δ it remains unchanged after a legal move.

Notice that $F(\mathbf{I}) = +1$ since $\mathbf{I}$ is an even permutation and since the blank square has not moved.

Thus, if a sequence of legal moves changes a given configuration into the solved configuration, the F value of the given configuration must be +1. Equivalently, if the given configuration has an F value of -1, the configuration cannot be converted into the solved configuration by legal moves.

Problem B You are handed a sliding block puzzle with numbered blocks in the following pattern. The lower right hand space does not have a block. You are asked to solve the puzzle. Explain why this cannot be done.

1	2	3	4
5	6	7	8
12	9	10	11
13	14	15	

Answer: The symmetry to obtain this configuration from the solved configuration is:

$$g = [9\ 10\ 11\ 12] = [9\ 10][10\ 11][11\ 12].$$

g is an odd permutation and so $\varepsilon(g) = -1$. Since the blank square is in location 16, $\delta(g) = 1$. Thus, $F(g) = -1$. By Problem A, this configuration is not solvable.

Problem C Explain why it is impossible to decorate a regular decagon so that the decorated decagon has exactly 8 symmetries.

Answer: A regular decagon has 20 symmetries. The symmetries of a decorated decagon are a subgroup of these. LaGrange's theorem implies that the number of symmetries of a decorated decagon must divide into 20. The number 8 is not a divisor of 20, so no such decorated decagon can exist.

Problem D Consider the subgroup $H = \{\mathbf{I}, R_{120}, R_{240}\}$ of D_6. List all the cosets of H in D_6.

Answer: By LaGrange's theorem, there will be 4 cosets. One of these cosets is H. To find another one, consider the coset $R_{60}H$. The symmetries in this coset are:

$$\begin{aligned} R_{60} \circ \mathbf{I} &= R_{60} \\ R_{60} \circ R_{120} &= R_{180} \\ R_{60} \circ R_{240} &= R_{300} \end{aligned}$$

A third coset can be obtained by using the reflection a:

$$\begin{aligned} a \circ \mathbf{I} &= a \\ a \circ R_{120} &= e \\ a \circ R_{240} &= c \end{aligned}$$

The fourth coset can be obtained by using the reflection b:

$$\begin{aligned} b \circ \mathbf{I} &= b \\ b \circ R_{120} &= f \\ b \circ R_{240} &= d \end{aligned}$$

Notice that each symmetry of D_6 is in one of the cosets we have listed.

Problem E Think of $g = [123][456]$ as a symmetry in $\mathbb{S}_6$.

Answer: Since $g^3 = \mathbf{I}$, $H = \langle g \rangle$ contains 3 symmetries.

Notice that $g = [12][23][45][56]$. Since g can be written as an even number of transpositions and is thus in $\mathbb{A}_6$. Each combination of g with itself is an even transposition, so H is a subgroup of $\mathbb{A}_6$. $\mathbb{A}_6$ is a subgroup of $\mathbb{S}_6$, so H is also.

Since $\mathbb{A}_6$ has $6!/2 = 360$ symmetries and H has 3 symmetries, by LaGrange's theorem, the number of cosets of H in $\mathbb{A}_6$ is $360/3 = 120$.

Since $\mathbb{S}_6$ has $6! = 720$ symmetries and H has 3 symmetries, by LaGrange's theorem, the number of cosets of H in $\mathbb{S}_6$ is $720/3 = 240$.

Problem Set 6

Problem 1 An n-dimensional cube is formed from an $(n-1)$ dimensional cube by moving it in a perpendicular direction and keeping track of the motion it makes. Thus, there is an $n-1$ cube at the start of the motion and one at the end. The motion itself does not have any vertices, so an n cube has twice the number of vertices of an $(n-1)$ cube. Since a 1-dimensional cube has 2 vertices, an n dimensional cube has 2^n vertices.

Problem 2: We use the same reasoning as in problem 1. First for a 4D cube. A 4D cube is formed by moving a 3D cube in a new direction. A 3D cube has 12 edges. Thus, a 4D cube will have the 12 edges of the starting cube, the 12 edges of the ending cube, and 8 new edges coming from moving the vertices. This brings the total number of vertices to 32. The 3D cube has 6 squares, and so the 4D cube will have 6 squares from the starting 3D cube, 6 from the ending 3D cube, and an additional 12 coming from the moving vertices. This means a 4D cube has a total of 24 squares. Similarly, a 4D cube will have 1 + 1 + 6 = 8 3D cubes because of the starting 3D cube, the ending 3D cube, and the 6 moving faces.

A 5D cube is formed by moving a 4D cube, so using reasoning similar to the above we get:

$$\begin{array}{rcll} 32+32+16 &=& 80 & \text{edges} \\ 24+24+32 &=& 80 & \text{squares} \\ 8+8+24 &=& 40 & \text{3D cubes} \\ 1+1+8 &=& 10 & \text{4D cubes} \end{array}$$

As far as volume goes you have to decide what that means. One reasonable answer is that it is the 5 dimensional analogue of what length is in 1 dimension, area is in 2 dimensions, and volume is in 3 dimensions. Using that analogy, the 5 dimensional volume of a 5 dimensional cube with edges of length 3 is 3^5.

Problem 3: The formula for a cone in 3 dimensions is

$$x_3 = \sqrt{x_1^2 + x_2^2}.$$

The formula for a cone is 4 dimensions is

$$x_4 = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

The slice of this 4D cone with $x_3 = 1$ has formula

$$x_4 = \sqrt{x_1^2 + x_2^2 + 1^2}.$$

This can be graphed on a computer using the equation:

$$z = \sqrt{x^2 + y^2 + 1}.$$