PS 7 Solutions

Abelard and Heloise are playing games of "toss the toast". If the toast lands butter side up, Heloise wins. If the toast lands butter side down, Abelard wins. Unbeknownst to Heloise, Abelard has used lard instead of butter, making it more likely for the toast to land butter side down. In fact, the probability that the toast lands butter side down is 2/3.

(1) Explain what it means to say that the probability that the toast lands butter side down is 2/3.

Solution: If the toast is tossed a large number of times, then approximately 2/3 of the time the toast lands butter side down.

(2) Assuming that the results of future tosses don't depend on the result of past tosses, what is the probability that Abelard will win 5 tosses in a row?

Solution: $\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot = \frac{2^5}{3^5} \approx .13$

(3) Assuming that the results of future tosses don't depend on the result of past tosses, out of 5 tosses, what is the probability that Heloise will win exactly twice?

Solution: There are "5 choose 2" ways for Heloise to win exactly 2 out of 5 tosses. "5 choose 2" = 10. Thus there are exactly 10 ways for Heloise to win exactly 2 out of 5 tosses. An example of such an outcome is "HHAAA" where H means that Heloise wins the toss and A means that Abelard wins the toss. Abelard has a 2/3 chance of winning the toss and Heloise has 1/3 chance of winning the toss. Thus $P(HHAAA) = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = 8/243 \approx .033$. Since in all the outcomes in which Heloise wins exactly two tosses there are 2 Hs and 3 As, the probability of each outcome in the event that Heloise wins twice is approximately .033. The probability of the event is the sum of the probabilities of the outcome in the event and so the probability that Heloise wins exactly 2 out of the 3 tosses is $10 \cdot .033 = .33$.

(4) Assume that the results of future tosses don't depend on the result of past tosses. Abelard and Heloise decide to play a game where the person to win the most tosses out of five tosses will win \$100. They toss the toast 3 times and Heloise is ahead 2 to 1. Unfortunately, their dog, Ignatius, then eats the toast. What is the probability that Heloise would have won the \$100?

Solution: There are four ways in which the game might finish: $\{HH, HA, AH, AA\}$. The probabilities of these outcomes in order are 1/9, 2/9, 2/9, and 4/9. The event that Heloise wins the game is $\{HH, HA, AH\}$. Thus, the probability that Heloise wins is 1/9 + 2/9 + 2/9 = 5/9.

(5) This problem continues the train of thought from the previous problem. What if instead of playing "the best out of 5 tosses", they played "first person to 3 wins"? (That is, as soon as someone wins 3 tosses, they stop playing) What is the probability that Heloise would have won the game? Be sure to explain your answer in relation to the previous problem.

Solution: It is still 5/9. In this case, the possible outcomes for concluding the game are $\{H, AH, AA\}$. The first has probability 1/3, the second probability 2/9, and the third probability 4/9. Thus the probability that Heloise wins is 1/3 + 2/9 = 5/9. Alternatively, whether or not Heloise wins on the fourth toss should not depend on whether or not the toast is tossed a fifth time.

(6) Abelard and Heloise find another piece of toast which Abelard again butters with lard. They decide to play the following game: They will toss the toast. If the toast lands butter side up, Abelard will give Heloise \$99. If the toast lands butter side down, Heloise will pay Abelard \$30. If they play this game many times, how much on average will Abelard win or lose? How much on average will Heloise win or lose?

Solution: The expected value for Abelard is

$$(-99)(1/3) + (30)(2/3) = -13.$$

Thus, on average, Abelard will lose \$13. Likewise, on average Heloise will win \$13.

(7) Heloise discovers that Abelard has buttered the toast with lard and demands that they switch to using a fair coin. (The probability of heads is 1/2 and the probability of tails is 1/2). Abelard and Heloise tossed the coin 15 times. Heloise always called Heads and Abelard always called Tails. Abelard wins the first 5 tosses. What is the probability that Abelard won 10 out of the 15 tosses?

Solution: Let F be the event that Abelard wins the first 5 tosses. Let E be the event that Abelard wins 10 out of 15 tosses. We are looking for $P(E|F)$. The formula for conditional probability says that:

$$P(E|F) = \frac{P(E \cap F)}{P(F)}.$$

Let's first calculate $P(F)$. The probability that Abelard wins a toss is $1/2$. Thus, the probability that he wins 5 tosses in a row is $(1/2)^5 = 1/32$.

The event $E \cap F$ consists of all outcomes where Abelard wins the first 5 tosses AND where Abelard wins 10 out of the 15 tosses. We first calculate the number of outcomes in $E \cap F$. Consider the 15 tosses

$$\begin{array}{ccccc} \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} \end{array}$$

If an outcome is in $E \cap F$ Abelard must win the first 5 tosses:

$$\begin{array}{ccccc} \underline{\;A\;} & \underline{\;A\;} & \underline{\;A\;} & \underline{\;A\;} & \underline{\;A\;} \\ \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} \end{array}$$

To be in $E \cap F$ exactly 5 of the remaining 10 tosses must go to Abelard. Thus $E \cap F$ has "10 choose 5" $= 252$ outcomes. Since there are 2^{15} possible outcomes

$$P(E \cap F) = \frac{252}{2^{15}} \approx .00769.$$

Plugging the answers for $P(F)$ and $P(E\cap F)$ into the formula for $P(E|F)$ we obtain

$$P(E|F) = \frac{252/2^{15}}{1/2^5} = \frac{252}{2^{10}} \approx .246.$$

(8) They decide to do this again. They toss the coin 15 times and Heloise wins 10 out of 15 tosses. What is the probability that Heloise won the first toss?

Solution: Let E be the event that Heloise wins exactly 10 out of 15 tosses. Let F be the event that Heloise wins the first toss. We wish to know $P(F|E)$. The formula tells us that

$$P(F|E) = \frac{P(F\cap E)}{P(E)}.$$

To be in $F\cap E$, Heloise must win the 1st toss and 9 out of the remaining 14 tosses. Thus, $F\cap E$ contains "14 choose 9" outcomes. Since there are 2^{15} possible outcomes

$$P(F\cap E) = \frac{14 \text{ choose } 9}{2^{15}}.$$

There are exactly "15 choose 10" ways for Heloise to win 10 out of 15 tosses. Thus,

$$P(E) = \frac{15 \text{ choose } 10}{2^{15}}.$$

Plugging into the formula we get:

$$P(F|E) = \frac{(14 \text{ choose } 9)/2^{15}}{(15 \text{ choose } 10)/2^{15}} = \frac{14 \text{ choose } 9}{15 \text{ choose } 10} = 2/3.$$