MA 302: Practice Exam 3 **Name:**________________________

The final exam is cumulative. The following problems concern only material since Exam 2. You should also use previous exams, practice exams, quizzes, and homework to study.

(1) Find a parameterization of the surface formed by the graph of $z = x^2 - y^2$ with (x, y) in the triangle in the xy-plane formed by the x-axis, the y-axis, and the line $y = -x + 1$.

Solution: How about:

$$\mathbf{X}(s,t) = \begin{pmatrix} s \\ t \\ s^2 - t^2 \end{pmatrix}$$

with $0 \leq s \leq 1$ and $0 \leq t \leq -s + 1$?

(2) Is the surface in the previous problem a smooth surface? If no, at what points is it not smooth?

Solution: The answer depends (somewhat) on your parameterization. The answer here is based on the parameterization above.

You can calculate that

$$\begin{aligned} \mathbf{T}_s &= (1, 0, 2s) \\ \mathbf{T}_t &= (0, 1, -2t) \\ \mathbf{N} &= (-2s, 2t, 1) \end{aligned}$$

Since $\mathbf{N}$ is never $\mathbf{0}$, and since $\mathbf{X}$ is obviously C^1, $\mathbf{X}$ is a smooth surface.

(3) Find a parameterization of the surface formed by rotating the curve $\begin{pmatrix} \cos t + 5 \\ 2\sin t \end{pmatrix}$ with $0 \leq t \leq 2\pi$ around the y-axis.

Solution: How about

$$\mathbf{X}(s,t) = \begin{pmatrix} \cos s(\cos t + 5) \\ 2\sin t \\ \sin s(\cos t + 5) \end{pmatrix}?$$

(4) Consider the surface

$$\mathbf{X}(s,t) = \begin{pmatrix} 2\sin 3t + t \\ \cos 2s \\ t^2 + s^2 \end{pmatrix}, \quad 0 \le t \le \pi/4, \quad 0 \le s \le \pi$$

Find the tangent and normal vectors to $\mathbf{X}$ at the point $(\pi/6, \pi/6)$. Is the surface smooth?

Solution:

We have

$$\begin{aligned} \mathbf{T}_s &= (0, -2\sin 2s, 2s) \\ \mathbf{T}_t &= (6\cos(3t) + 1, 0, 2t) \\ \mathbf{N} &= (-4t\sin 2s, 2s(6\cos 3t + 1), 2\sin 2s(6\cos 3t + 1) \end{aligned}$$

Plug $(\pi/6, \pi/6)$ into the above equations to get:

$$\begin{aligned} \mathbf{T}_s &= (0, -\sqrt{3}, \pi/3) \\ \mathbf{T}_t &= (1, 0, \pi/3) \\ \mathbf{N} &= (-\pi\sqrt{3}/3, \pi/3, \sqrt{3}) \end{aligned}$$

Since $\mathbf{N}(\pi/6, \pi/6) \neq \mathbf{0}$, the surface is smooth at that point.

(5) Suppose that $\mathbf{F}\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a C^1 vector field, and that $\mathbf{X}\colon D \to \mathbb{R}^3$ is a smooth, oriented surface. Let $h\colon E \to D$ be a smooth, orientation reversing change-of coordinate function. Prove that

$$\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} = -\iint_{\mathbf{X}\circ h} \mathbf{F} \cdot d\mathbf{S}.$$

Solution: See your course notes or adapt the solution to the next problem.

(6) Suppose that $f\colon \mathbb{R}^3 \to \mathbb{R}$ is a C^1 vector field, and that $\mathbf{X}\colon D \to \mathbb{R}^3$ is a smooth, oriented surface. Let $h\colon E \to D$ be a smooth change-of coordinate function. Prove that

$$\iint_{\mathbf{X}} f\, dS = \iint_{\mathbf{X}\circ h} f\, dS.$$

Solution: By definition,

$$\iint_{\mathbf{X}\circ h} f\, dS = \iint_E f(\mathbf{X}\circ h)||\mathbf{N}||\, dA$$

Let $\mathbf{Y} = \mathbf{X} \circ h$. It is a fact (proved in class) that $\mathbf{N}_\mathbf{Y} = (\det Dh)\mathbf{N}_\mathbf{X} \circ h$. Thus,

$$\iint_{\mathbf{X}\circ h} f\, dS = \iint_E f(\mathbf{X}\circ h)||\mathbf{N}_\mathbf{X} \circ h||\,|\det Dh|\, dA$$

By the change of coordinates theorem, this give us:

$$\iint_{\mathbf{X}\circ h} f\, dS = \iint_E f(\mathbf{X})||\mathbf{N}_\mathbf{X}||\, dA$$

By the definition of surface integral we then get our result:

$$\iint_{\mathbf{X}\circ h} f\, dS = \iint_\mathbf{X} f\, dS.$$

(7) Suppose that $\mathbf{X}\colon\ D \to \mathbb{R}^3$ is a smooth, oriented surface with unit normal $\mathbf{n}$. Suppose that $\mathbf{F}\colon\ \mathbb{R}^3 \to \mathbb{R}^3$ is a C^1 vector field. Prove that

$$\iint_\mathbf{X} \mathbf{F}\cdot d\mathbf{S} = \iint_\mathbf{X} \mathbf{F}\cdot\mathbf{n}\, dS.$$

Solution: We have $\mathbf{n} = \mathbf{N}/||\mathbf{N}||$. Thus,

$$\begin{aligned}
\textstyle\iint_\mathbf{X} \mathbf{F}\cdot d\mathbf{S} &= \textstyle\iint_D (\mathbf{F}\circ\mathbf{X})\cdot\mathbf{N}\, dA \\
&= \textstyle\iint_D (\mathbf{F}\circ\mathbf{X})\cdot(||\mathbf{N}||\mathbf{n})\, dA \\
&= \textstyle\iint_D (\mathbf{F}\circ\mathbf{X})\cdot\mathbf{n}\,||\mathbf{N}||dA \\
&= \textstyle\iint_\mathbf{X} \mathbf{F}\cdot\mathbf{n}\, dS.
\end{aligned}$$

(8) Use the previous result to integrate the vector field $\mathbf{F}(x,y,z) = (x,y,z)$ over the unit sphere (with outward normal) in $\mathbb{R}^3$.

Solution: At a point (x,y,z) on the unit sphere S, there is the normal $\mathbf{n} = (x,y,z)$. Thus, $\mathbf{F}\cdot\mathbf{n} = x^2+y^2+z^2$. Since (x,y,z) is on the unit sphere, $\mathbf{F}\cdot\mathbf{n} = 1$. Thus,

$$\iint_S \mathbf{F}\, dS = \iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_S 1\, dS.$$

This last quantity is just the surface area of the sphere, which is 4π.

(9) Let S be the disc of radius 1 centered at $(1,0,0)$ in $\mathbb{R}^3$ which is parallel to the yz-plane. Orient S with normal vector pointing in the direction of the postive x-axis. Use the definition of surface integral to calculate the flux of $\mathbf{F}(x,y,z) = (-xy, yz, xz)$ through S.

Solution: Parameterize S as:

$$\mathbf{X}(s,t) = \begin{pmatrix} 1 \\ s \\ t \end{pmatrix}$$

with (s,t) in the region D defined by $0 \leq s^2 + t^2 \leq 1$. It is easy to calculate $\mathbf{N} = (1,0,0)$. Then,

$$\mathbf{F} \cdot \mathbf{N}(x,y,z) = -xy.$$

Thus, by the definition of surface integral, the flux of $\mathbf{F}$ through S is

$$\iint_D \mathbf{F} \cdot \mathbf{N}(\mathbf{X}(s,t))\, dA = \iint_D -s\, ds\, dt.$$

Change to polar coordinates by setting $s = r\cos\theta$ and $t = r\sin\theta$. Then the integral above is equal to (by the change of coordinates theorem):

$$\int_0^1 \int_0^{2\pi} -r^2 \cos\theta\, d\theta dr$$

Since $\int_0^{2\pi} \cos\theta d\theta = 0$, the flux equals 0.

(10) Use the same surface S and $\mathbf{F}$ as in the previous problem, but now use Stoke's theorem to calculate the flux of the curl of the previous problem.

Solution: By Stoke's theorem,

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F}\, d\mathbf{s}.$$

Parameterize ∂S as:

$$\mathbf{x}(t) = \begin{pmatrix} 1 \\ \cos t \\ \sin t \end{pmatrix}$$

with $0 \leq t \leq 2\pi$.

Notice that $\mathbf{x}$ gives ∂S the orientation induced by the orientation on S. Then,

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} \mathbf{F}(\mathbf{x})(t) \cdot \mathbf{x}'(t)\, dt.$$

Calculations show that this equals

$$\begin{aligned} \textstyle\int_0^{2\pi} -\cos t \sin^2 t + \sin t \cos t\, dt &= \textstyle\int_0^{2\pi} -\cos t \sin^2 t\, dt + \int_0^{2\pi} \sin t \cos t\, dt \\ &= 0. \end{aligned}$$

(11) Let $S \subset \mathbb{R}^3$ be an ellipsoid enclosing the origin, oriented outward. Let $P \subset \mathbb{R}^3$ be a cube enclosing the origin and enclosed by S. Orient P outward. Let $\mathbf{F}$ be an incompressible vector field defined on $\mathbb{R}^3 - \{\mathbf{0}\}$. Prove that the flux of $\mathbf{F}$ through P is the same as the flux of $\mathbf{F}$ through S.

Solution: Let V be the region between S and P. Orient ∂V with a unit normal that points out of V. Then by the divergence theorem:

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} - \iint_P \mathbf{F}\cdot d\mathbf{S} &= \iint_{\partial V} \mathbf{F}\cdot d\mathbf{S} \\ &= \iiint_V \operatorname{div}\mathbf{F}\,dV \\ &= \iiint_V 0\,dV \\ &= 0.\end{aligned}$$

Consequently, $\iint_S \mathbf{F}\cdot d\mathbf{S}$ equals $\iint_P \mathbf{F}\cdot d\mathbf{S}$.

(12) Let $\mathbf{F}\colon \mathbb{R}^3 \to \mathbb{R}^3$. Let $\mathbf{a}$ be a point in $\mathbb{R}^3$. For each $n \in \mathbb{N}$, let V_n be a compact 3-dimensional region containing $\mathbf{a}$, such that the regions V_n limit to $\mathbf{a}$. Oriente the boundary of V_n outwards. Use the divergence theorem to prove that

$$\operatorname{div}\mathbf{F}(\mathbf{a}) = \lim_{n\to\infty} \frac{1}{\operatorname{vol} V_n} \iint_{\partial V_n} \mathbf{F}\cdot d\mathbf{S}.$$

Solution: Suppose that n is large enough so that $\mathbf{F}(\mathbf{x}) \approx \mathbf{F}(\mathbf{a})$ for all $\mathbf{x} \in V_n$. Then, by the divergence theorem:

$$\begin{aligned}\iint_{\partial V_n} \mathbf{F}\cdot d\mathbf{S} &= \iiint_{V_n} \operatorname{div}\mathbf{F}\,dV \\ &\approx \iiint_{V_n} \operatorname{div}\mathbf{F}(\mathbf{a})\,dV \\ &= \operatorname{div}\mathbf{F}(\mathbf{a}) \iiint_{V_n} dV \\ &= \operatorname{div}\mathbf{F}(\mathbf{a})(\operatorname{vol} V_n).\end{aligned}$$

That is,

$$\operatorname{div}\mathbf{F}(\mathbf{a}) \approx \frac{1}{\operatorname{vol} V_n} \iint_{\partial V_n} \mathbf{F}\cdot d\mathbf{S}.$$

As $n \to \infty$ this approximation becomes exact, proving the result.

(Note: This proof is actually non-rigorous. To make it rigorous we would need to use the mean value theorem for integrals.)

(13) Let S be the box with corners $(\pm 1, \pm 1, \pm 1)$, oriented outward. Let $\mathbf{F}(x,y,z) = \begin{pmatrix} xyz \\ xy \\ z \end{pmatrix}$. Find the flux of $\mathbf{F}$ through S.

Solution: Use the divergence theorem. We have $\operatorname{div}\vec{F}(x,y,z) = yz + x + 1$. The divergence says the flux through S is equal to

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} yz + x + 1\,dx\,dy\,dz = 8.$$

(14) Let S be a surface formed by rotating the image of $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ \sin t \end{pmatrix}$, $2\pi \leq t \leq 3\pi$ around the y–axis. Orient S so that at the point $(2\pi + \pi/2, 1, 0)$ there is an upward pointing normal vector. For the following vector fields, find the flux of the vector field through S. (Hint: there are easy ways and there are hard ways...)

For all of the solutions below, let A be the annulus in the xz-plane with the same boundary as S and oriented upward. Let V be the region between A and S.

(a) $\mathbf{F}(x,y,z) = \begin{pmatrix} x+y \\ -y+z \\ -x+y \end{pmatrix}$

Solution:

We have by the divergence theorem, that

$$\iint_S \mathbf{F} \cdot d\mathbf{S} - \iint_A \mathbf{F} \cdot d\mathbf{S} = \iiint_V \operatorname{div} \mathbf{F} \, dV.$$

The divergence of $\mathbf{F}$ is 0, so $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_A \mathbf{F} \cdot d\mathbf{S}$.

Parameterize A as:

$$\mathbf{X}(s,t) = \begin{pmatrix} t\cos s \\ 0 \\ t \sin s \end{pmatrix}$$

for $2\pi \leq t \leq 3\pi$ and $0 \leq s \leq 2\pi$. Calculate:

$$\mathbf{N} = \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix}$$

Notice that this gives A the correct orientation.

Now, $\mathbf{F} \cdot \mathbf{N}(s,t) = t^2 \sin s$. Thus, the flux through A is

$$\int_0^{2\pi} \int_{2\pi}^{3\pi} t^2 \sin s \, dt ds = 0.$$

(b) $\mathbf{F}(x,y,z) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Solution: For this problem you can either use Stokes' theorem or the method of the previous part. In this case

$$\iint_A \mathbf{F} \cdot d\mathbf{S} = \iint_A \mathbf{F} \cdot \mathbf{n}\, dS = \iint_A dS = 5\pi^3.$$

(c) $\mathbf{F}(x,y,z) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Solution: Once again the flux through S equals the flux through A, and so since $\mathbf{F}$ is tangent to A, the flux through A is zero.

(d) $\mathbf{F}(x,y,z) = \frac{1}{x^2+z^2} \begin{pmatrix} -z \\ 0 \\ x \end{pmatrix}$

Solution: In this case, recall that the flow lines for $\mathbf{F}$ are circles centered at the origin parallel to the xz-plane. Consequently, $\mathbf{F}$ is tangent to S and so the flux through S is zero.

(15) Prove that inside a hollow planet there is no gravity. (You may use Gauss' Law of Gravitation.)

Solution: See class notes. A solution will be posted here later.