

# On Spectral Theorem

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# Chapter 1

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# Chapter 2

## Abstract

### 2.1 Abstract

There are many instances where the theory of eigenvalues and eigenvectors has its applications. However, Matrix theory, which usually deals with vector spaces with finite dimensions, also has its constraints. Spectral theory, on the other hand, generalizes the ideas of eigenvalues and eigenvectors and applies them to vector spaces with arbitrary dimensions. In the following chapters, we will learn the basics of spectral theory and in particular, we will focus on one of the most important theorems in spectral theory, namely the spectral theorem. There are many different formulations of the spectral theorem and they convey the "same" idea. In this thesis, we are going to see two of the approaches toward the theorem. The intention is that the more aspects you know about the theorem as a tool, the more powerful you are with that tool. Chapter 3,4,5,6 will be devoted to the measure theoretical approach of the spectral theorem and the last chapter will focus on the algebraic approach of the theorem.

The measure theoretical approach needs 3 important building blocks which will be discussed separately . In chapter 3, we will focus on the Stone-Weierstrass theorem based on materials from [3], and [9]. In chapter 4, we will talk about the Riesz Representation theorem based on materials from [8], and [4]. (Note: this chapter requires basic measure theory background which can be

found in [10]). In chapter 5, we will discuss properties of Spectral Radius based on contents in [8] and [7]. (Note: basic functional analysis knowledge from [7] and [6] is assumed). Once we have equipped with the 3 building blocks, we can proceed and try to gain understandings about the spectral theorem and what the theorem says. Chapter 6 is an expository article based on the paper by P.R. Halmos[5]. Halmos provides a clear picture of the measure theoretical approach toward the spectral theorem. This approach provides a way to see the theorem with concrete "objects".

The algebraic approach, when compared to the measure theoretical approach, is simpler and more concise. However, the algebraic approach is more abstract and more difficult for one to gain intuitions from the formulation of the ideas. In chapter 7, based on materials from [2], and [1], we will introduce Banach algebra and then discuss a particular kind of Banach algebra where we can rediscover the spectral theorem.

I hope that after reading the chapters, one can gain a concrete understanding of the spectral theorem and learn two ways to see it as an important result of spectral theory. Maybe through this article, one can further develop the notion of connectedness between different mathematical fields across a broad range.

# Chapter 3

## Stone Weierstrass Theorem

### 3.1 Introduction

As an important result that we learned in real analysis, the Weierstrass Approximate theorem states that we can approximate any continuous function on a compact interval, to an arbitrary degree, with polynomials. However, the Weierstrass Approximate theorem only applies to continuous functions from  $\mathbb{R}$  to  $\mathbb{R}$ . The Stone Weierstrass theorem, on the other hand, is a generalization of the Weierstrass Approximate theorem. The Stone Weierstrass theorem applies to any continuous function with a domain of a compact Hausdorff space  $K$  and a target space of  $\mathbb{R}$  or  $\mathbb{C}$ . And instead of just polynomials, we can approximate continuous functions with elements of a subalgebra of continuous functions from  $K$  to  $\mathbb{R}$  or  $\mathbb{C}$ . In this section, we want to investigate Stone Weierstrass theorem in details.

### 3.2 Preliminary Knowledge

Before we move on to the proof of the Stone Weierstrass theorem, we need to know the following definitions.

**Definition 3.2.1.** (Hausdorff) A topological space  $X$  is called Hausdorff if for any  $x, y \in X, x \neq y$ , there exists open sets  $U_x, U_y$  such that  $x \in U_x, y \in U_y$  and  $U_x \cap U_y = \emptyset$ .

**Definition 3.2.2.**  $C(X, \mathbb{R})$  denotes the space of continuous functions from  $X$  to  $\mathbb{R}$ .

For  $f, g \in C(X, \mathbb{R})$ , we let  $fg$  be defined as  $fg(x) = f(x)g(x)$  for all  $x \in X$ .

**Definition 3.2.3.** (Sup Norm) the sup norm assigns to real- or complex-valued bounded functions  $f$  defined on a set  $X$  the non-negative number  $\|f\| = \sup\{|f(x)| : x \in X\}$ .

Given the fact that we only consider continuous functions  $f$  from a compact space  $X$  to  $\mathbb{R}$ , the supremum  $\sup\{|f(x)| : x \in X\}$  always exists. As a matter of fact, when equipped with the sup norm,  $C(X, \mathbb{R})$  forms a normed space.

**Definition 3.2.4.** (Algebra) A subset  $A$  of  $C(X, \mathbb{R})$  is an algebra if it is a subspace of  $C(X)$  that is closed under multiplication (ie: if  $f, g \in C(X, \mathbb{R})$ , then  $fg \in C(X, \mathbb{R})$ ).

For  $f, g \in C(X, \mathbb{R})$ , we define  $f \vee g(x) = \max\{f(x), g(x)\}$  and  $f \wedge g(x) = \min\{f(x), g(x)\}$ .

**Definition 3.2.5.** (Vector Lattice) A subset  $L$  of  $C(X, \mathbb{R})$  is a vector lattice if it is a subspace that is closed under these operations, that is, if  $f, g \in L$  then  $f \vee g, f \wedge g \in L$ .

Observe that  $f \vee g = \frac{1}{2}(f + g) + \frac{1}{2}|f - g|$  and  $f \wedge g = \frac{1}{2}(f + g) - \frac{1}{2}|f - g|$ . On the other hand  $|f| = f \vee -f$ . Thus, we may conclude that an algebra  $A$  is a vector lattice if and only if for any  $f \in A, |f| \in A$ .

**Definition 3.2.6.** (Separate points and Vanish at  $x$ ) A set  $S$  of functions on  $X$  separate points if for each pair of points  $x, y \in X$ , there is a function  $f \in S$  such that  $f(x) \neq f(y)$ . We say that  $S$  vanishes at  $x \in X$  if  $f(x) = 0$  for all  $f \in S$ .

Intuitively speaking, to approximate any continuous function on  $X$  with functions in  $A$ , then  $A$  should separate points because for any  $x, y \in X$  we can always find continuous functions that take different values on  $x, y$ . Also,  $A$  must vanish at no point because otherwise, we cannot approximate the constant 1 function. We will now proceed to the statement of the Stone Weierstrass theorem and its proof.



### 3.3 Stone Weierstrass theorem

Firstly we consider the Weierstrass Approximation theorem and then we will go to its generalized version which is the Stone Weierstrass theorem. Interestingly, in order to prove the Stone Weierstrass theorem, we will actually use the result of the Weierstrass Approximation theorem.

**Theorem 3.3.1.** (*Weierstrass Approximation theorem*) Suppose  $f \in C([0, 1], \mathbb{R})$ ,  $\varepsilon > 0$ , then there exists polynomial  $p$  on  $[0, 1]$  such that  $\|f - p\| \leq \varepsilon$ .

The following proof is due to Bernstein.

*Proof.* We define Bernstein polynomials on  $[0, 1]$  as follows:

$$B_{n,k} = \binom{n}{k} x^k (1-x)^{n-k}; k = 0, 1, \dots, n.$$

Observe that  $B_{n,k}$  is always non-negative. Using Binomial Formula, we have that

$$\sum_{k=0}^n B_{n,k}(x) = 1 \tag{3.1}$$

Then, with some algebraic manipulations, it is easy to check that

$$\sum_{k=0}^n k B_{n,k}(x) = nx \tag{3.2}$$

$$\sum_{k=0}^n k(k-1) B_{n,k}(x) = n(n-1)x^2 \tag{3.3}$$

Then using the 3 identities above, with some more algebraic manipulation, we may also conclude:

$$\sum_{k=0}^n \frac{k}{n} B_{n,k}(x) = x \tag{3.4}$$

$$\sum_{k=0}^n \left(\frac{k}{n} - x\right)^2 B_{n,k}(x) = \frac{x(1-x)}{n}. \tag{3.5}$$

Now, for  $n \in \mathbb{N}$ , define the polynomial

$$B_n(f) = \sum_{k=0}^n f\left(\frac{k}{n}\right) B_{n,k}(x).$$

Claim: there exists  $N \in \mathbb{N}$  such that for all  $n > N$ ,  $\|B_n(f) - f\| < \varepsilon$ .

Since  $f$  is continuous on a compact set, then  $f$  is uniformly continuous. Then there exists  $\delta > 0$  such that  $|f(x) - f(y)| < \frac{\varepsilon}{2}$  whenever  $|x - y| < \delta$ . Notice that using identity (1), we have

$$B_n(f)(x) - f(x) = \sum_{k=0}^n \left[ f\left(\frac{k}{n}\right) - f(x) \right] B_{n,k}(x)$$

Therefore if  $|x - \frac{k}{n}| < \delta$ , since  $B_{n,k}$  is always non-negative,

$$|B_n(f)(x) - f(x)| \leq \sum_{k=0}^n \left| f\left(\frac{k}{n}\right) - f(x) \right| B_{n,k}(x) < \frac{\varepsilon}{2} \sum_{k=0}^n B_{n,k}(x) = \frac{\varepsilon}{2} < \varepsilon.$$

If  $|x - \frac{k}{n}| \geq \delta$ , then using identity (5)

$$\sum_{k=0}^n B_{n,k}(x) = \sum_{k=0}^n \frac{\left(\frac{k}{n} - x\right)^2}{\left(\frac{k}{n} - x\right)^2} B_{n,k}(x) \leq \frac{1}{\delta^2} \sum_{k=0}^n \left(\frac{k}{n} - x\right)^2 B_{n,k}(x) = \frac{1}{\delta^2} \frac{x(1-x)}{n}$$

Then, since  $x(1-x)$  has a maximum value of  $\frac{1}{4}$  on  $[0, 1]$ , we can conclude that

$$|B_n(f)(x) - f(x)| \leq \sum_{k=0}^n \left| f\left(\frac{k}{n}\right) - f(x) \right| B_{n,k}(x) \leq 2\|f\| \sum_{k=0}^n B_{n,k}(x) \leq \frac{\|f\|}{\delta^2 n}.$$

Then take  $n > \frac{\varepsilon \delta^2}{\|f\|}$ , we have that  $n > N$ ,  $\|B_n(f) - f\| < \varepsilon$ . This completes the proof.  $\square$

Now, we may proceed to the Stone Weierstrass theorem.

**Theorem 3.3.2.** (Stone Weierstrass theorem) Suppose an algebra  $A$  of continuous real-valued functions on a compact Hausdorff space  $X$  separates points and does not vanish at any point, then  $A$  is dense in  $C(X, \mathbb{R})$ .

" $A$  is dense in  $C(X, \mathbb{R})$ " means that for all  $f \in C(X, \mathbb{R})$ , for all  $\varepsilon > 0$ , there exists  $g \in A$  such that  $\|f - g\|_X < \varepsilon$ .

Before we actually prove the theorem, there are two useful lemma that will be useful The following lemmas will be used in our proof of the Stone Weierstrass theorem.

**Lemma 3.3.3.** *If  $A$  is an algebra of real-valued continuous functions on  $X$ , then its closure  $\bar{A}$  is a closed algebra and a vector lattice.*

*Proof.* We show that  $A$  is an algebra first. Since the closure of a subspace is a subspace, then  $\bar{A}$  is a subspace. To see that  $\bar{A}$  is closed under point-wise multiplication, let  $f, g \in \bar{A}$  be given. Then there exists sequences  $f_n, g_n$  in  $A$  such that  $f_n$  uniformly converges to  $f$  and  $g_n$  uniformly converges to  $g$ . Since  $A$  is an algebra,  $f_n g_n$  is a sequence in  $A$ . Since for each  $n$ ,  $f_n, g_n$  are continuous functions on a compact set, then they are all bounded, say by  $M_n \in \mathbb{R}$ . Notice that since  $f, g$  are uniform limits of bounded functions, then  $f, g$  are both bounded, say by  $M \in \mathbb{R}$ . Then

$$\begin{aligned} \|f_n g_n - f g\|_X &\leq \|f_n g_n - f_n g + f_n g - f g\|_X \\ &\leq \|f_n g_n - f_n g\|_X + \|f_n g - f g\|_X \\ &\leq M_n \|g_n - g\|_X + M \|f_n - f\|_X \end{aligned}$$

Since  $\|g_n - g\|_X$  and  $\|f_n - f\|_X$  can be made arbitrarily small with big enough  $n$ , we have that  $\|f_n g_n - f g\|_X$  can be made arbitrarily small with big enough  $n$ . Thus  $f_n g_n$  uniformly converges to  $f g$ , so  $\bar{A}$  is closed under point-wise multiplication, hence  $\bar{A}$  is an algebra.

Now, we want to show that  $\bar{A}$  is a vector lattice. From the observation we made after definition of vector lattice, we know that  $A$  is a vector lattice iff for all  $f \in A$ ,  $|f| \in A$ . Thus, we will prove that  $A$  is closed under taking absolute value and we will be done.

Given  $f \in \bar{A}$ , let  $|t| = h(t)$  where  $t \in [-\|f\|_X, \|f\|_X]$ , then since  $h$  is a continuous function on a compact interval, we know from Weierstrass Approximate theorem that there is a sequence of polynomials  $p_n$  such that  $p_n$  uniformly converges to  $h$  on  $[-\|f\|_X, \|f\|_X]$ . Let  $q_n(t) = p_n(t) - p_n(0)$ , then  $q_n(0) = 0$  for all  $n$ . Notice that  $\|h - q_n\| \leq \|h - p_n\| + |h(0) - p_n(0)|$ . Since  $p_n$

uniformly converges to  $h$ , then RHS converges to 0, so  $q_n$  also uniformly converges to  $h$ . We will now show that  $|f| \in \bar{A}$ . If  $q$  is a polynomial on  $[-\|f\|_X, \|f\|_X]$ , and  $q(0) = 0$ , then  $q = a_1z + a_2z^2 + \dots + a_nz^n$  where  $z \in [-\|f\|_X, \|f\|_X]$ . Then  $q \circ f = a_1f + a_2f^2 + \dots + a_nf^n$ . Since  $\bar{A}$  is an algebra, then  $q \circ f \in \bar{A}$  too. Furthermore, if  $p, q$  are two such polynomials, then

$$\begin{aligned} \|p \circ f - q \circ f\|_X &= \sup\{|p(f(x)) - q(f(x))| : x \in X\} \\ &\leq \sup\{|p(z) - q(z)| : z \in [-\|f\|_X, \|f\|_X]\} = \|p - q\| \end{aligned}$$

Since  $q_n$  is Cauchy, and  $\|q_n \circ f - q_m \circ f\|_X \leq \|q_n - q_m\|$ , then  $q_n \circ f$  is Cauchy and its limit  $g$  belongs to  $\bar{A}$ . Observe  $g(x) = \lim_{n \rightarrow \infty} q_n \circ f(x) = h \circ f(x) = |f(x)|$ . Thus  $|f| \in \bar{A}$ . From previous observation we know that an algebra  $A$  is a vector lattice if and only if for any  $f \in A$ ,  $|f| \in A$ . Thus we have that  $\bar{A}$  is a vector lattice.  $\square$

**Lemma 3.3.4.** *Suppose  $A$  is an algebra of functions on a set  $X$ ,  $A$  separates points on  $X$ , and  $A$  vanishes at no point of  $X$ . Suppose  $x, y \in X$  and  $a, b$  are constants. Then  $A$  contains a function  $f$  such that  $f(x) = a$ ,  $f(y) = b$ .*

*Proof.* Since  $A$  separates points and vanishes at no point. Then there exists  $g, h, k \in A$  such that

$$g(x) \neq g(y), h(x) \neq 0, k(y) \neq 0$$

Let

$$s = gk - g(x)k, t = gh - g(y)h$$

Since  $A$  is an algebra, then  $u, v \in A$ . Notice that  $s(x) = 0$  and  $t(y) = 0$ ;  $s(y) \neq 0$  and  $t(x) \neq 0$ . Then consider the function  $f$  where

$$f = \frac{as}{s(x)} + \frac{bt}{t(y)}$$

Notice that  $f(x) = a$ ,  $f(y) = b$  so that  $f$  is our desired function.  $\square$

Equipped with these two lemmas, we are now ready to prove the Stone Weierstrass theorem.

*Proof.* Suppose  $f \in C(X, \mathbb{R})$  is given. Suppose  $\varepsilon > 0$ . We will find a function  $g \in \bar{A}$  such that  $\|f - g\| < \varepsilon$ .

For each pair of points  $x, y \in X$ , we can find a function  $g_{x,y} \in A$  such that  $g_{x,y}(x) = f(x)$  and  $g_{x,y}(y) = f(y)$  using the previous lemma.

Fix  $y$ . For each  $x \neq y$ , let  $U_x = \{z \in X : g_{x,y}(z) > f(z) - \varepsilon\} = (g_{x,y} - f)^{-1}(-\varepsilon, \infty)$ . Notice that  $U_x$  is open and  $U_x$  contains  $x, y$ . Then the set  $S = \{U_x : x \in X \setminus \{y\}\}$  forms an open cover of  $X$ . Since  $X$  is compact Hausdorff, we can find a finite subcover  $\{U_{x_1}, \dots, U_{x_n}\}$  of  $S$ .

Let  $g_y = g_{x_1,y} \vee \dots \vee g_{x_n,y}$ . We know that  $g_y \in \bar{A}$  by Lemma 3.2. Observe that for all  $x \in X$ , we have  $g_y(x) > f(x) - \varepsilon$ .

Now for each  $y$ , let  $V_y = \{z \in X : g_y(z) < f(z) + \varepsilon\} = (g_y - f)^{-1}(-\infty, \varepsilon)$ . Then  $T = \{V_y : y \in X\}$  forms an open cover of  $X$ . Since  $X$  is compact Hausdorff, we can find a finite subcover  $\{V_{y_1}, \dots, V_{y_n}\}$  of  $T$ .

Let  $g = g_{y_1} \wedge \dots \wedge g_{y_n}$ . Then for all  $x \in X$   $g(x) < f(x) + \varepsilon$ . Since for all  $1 \leq i \leq n$ , we have  $g_{y_i}(x) > f(x) - \varepsilon$  for all  $x \in X$ . Then we also have that  $g(x) > f(x) - \varepsilon$  for all  $x \in X$ . Thus  $\|g - f\| < \varepsilon$  as required.  $\square$

**Corollary 3.3.5.** *Suppose an algebra  $A$  of continuous complex-valued functions on a compact Hausdorff space  $X$  separates points and does not vanish at any point and  $A$  is closed under complex conjugation, then  $A$  is dense in  $C(X, \mathbb{C})$ .*

*Proof.* Since  $A$  is closed under complex conjugation, we may apply similar arguments to show this. Let  $R = \{Re(f) : f \in A\}$  (Note  $Im(f) = Re(-i \cdot f) \in R$  for all  $f \in A$ ).

I claim that  $R$  is a algebra that vanishes at no point and separate points.

Notice that  $Re(f) = \frac{f + \bar{f}}{2} \in R$  for all  $f$ . Thus  $R \subset A$ . Also notice that if we have two elements from  $R$ , say  $Re(f_1), Re(f_2)$ , then their product  $Re(f_1) \cdot Re(f_2) \in A$ , moreover,  $Re(f_1) \cdot Re(f_2)$  is a real valued function thus  $Re(f_1) \cdot Re(f_2) \in R$ . Hence  $R \subset A$  is a subalgebra and therefore an algebra of  $C(X, \mathbb{R})$ .

$R$  separate points since for any  $x, y \in X$ , there exists function  $f \in A$  such that  $f(x) \neq f(y)$ . This implies that either  $Re(f(x)) \neq Re(f(y))$  or  $Im(f(x)) \neq Im(f(y))$ . Hence  $R$  separates points.

$R$  vanishes at no point because  $A$  vanishes at no point and  $R \subset A$ .

Thus we have that  $R$  can approximate every real valued function on  $X$  uniformly.

Now, suppose  $h \in C(X, \mathbb{C})$ , then  $Re(h), Im(h)$  can be uniformly approximated by  $s, t \in R$ . Since

$$\|h - s + it\|_{\infty} \leq \|Re(h) - s\|_{\infty} + \|Im(h) - t\|_{\infty}$$

Then  $h$  can be uniformly approximated by elements in  $A$  and we are done. □

# Chapter 4

## Riesz Representation Theorem

### 4.1 Introduction

In this section, we are going to talk about the Riesz Representation Theorem. As a matter of fact, the Riesz Representation Theorem refers to a class of theorems. Generally speaking, the Riesz Representation Theorem describes the dual space of a Banach space. The most famous version of the Riesz Representation Theorem says that a Hilbert space and its dual space are isometrically conjugate-isomorphic to each other. However, in this section, we will be focusing on another version of Riesz Representation Theorem which will establish the relationship between the space of real-valued continuous functions defined on a compact Hausdorff space  $K$  (ie:  $C(K, \mathbb{R})$ ) and its dual. As always, before we actually prove the Riesz Representation Theorem, we need to firstly, go through the definitions required to understand the statement of the Riesz Representation Theorem. This section assumes basic measure theory knowledge.

## 4.2 Preliminary Knowledge

**Definition 4.2.1.** ( $\sigma$ -algebra) A  $\sigma$ -algebra on a set  $X$  is a collection  $\Omega$  of subsets of  $X$  that is closed under complement, countable union, and countable intersection. Also,  $\emptyset \in \Omega$  is required.

**Definition 4.2.2.** (Borel Sets) A Borel set  $B$  is any set in a topological space  $X$  that can be formed from open sets through the operations of countable union, countable intersection, and relative complement.

It can be proved that the collection of all Borel sets  $B_X$  in a topological space  $X$  forms a  $\sigma$ -algebra known as the **Borel algebra** and as a matter of fact, the Borel algebra is the smallest  $\sigma$ -algebra that contains all the open sets in  $X$ .

**Definition 4.2.3.** (Borel measure) A Borel measure is a measure  $\mu : B_X \rightarrow [0, \infty]$ .

**Definition 4.2.4.** (Regular) A Borel measure  $\mu$  on a space  $X$  is said to be regular if:

- (i.)  $\mu(A) = \inf\{\mu(O) : A \subset O, O \text{ is open}\}$
- (ii.)  $\mu(A) = \sup\{\mu(K) : K \subset A, K \text{ is compact}\}$

for all  $A \in B_X$ .

**Definition 4.2.5.** (Positive Linear Functional) A map  $l : C(X, \mathbb{R}) \rightarrow \mathbb{C}$  is said to be a positive linear functional if:

- (i.)  $l(\alpha x + \beta y) = \alpha l(x) + \beta l(y)$  for all  $\alpha, \beta \in \mathbb{C}$  and  $x, y \in C(X, \mathbb{R})$ .
- (ii.)  $l(\gamma) \geq 0$  for all  $\gamma \in C(X, \mathbb{R})$  such that  $\gamma(x) \geq 0$  for all  $x \in X$ .

## 4.3 Riesz Representation Theorem

We are now ready to prove the theorem.



**Theorem 4.3.1.** (*Riesz Representation Theorem for  $C(X, \mathbb{R})$* ) Let  $X$  be a compact metric space. If  $l : C(X, \mathbb{R}) \rightarrow \mathbb{C}$  is a positive bounded linear functional, then there exists a unique Borel measure such that

$$l(f) = \int f(x) d\mu(x)$$

And  $\mu$  is finite (ie:  $\mu(X) < \infty$ ).

*Proof.* For a given positive linear functional  $l$  on  $C(X, \mathbb{R})$  where  $X$  is a compact metric space, we will go to construct a Borel measure  $\mu$  that satisfies the properties in the statement. We start by considering the following map  $\mu^*$  defined on subsets of  $X$ .

For any open  $O \subset X$ , let

$$\mu^*(O) = \sup\{l(f) : f \in C(X, \mathbb{R}), 0 \leq f(x) \leq 1 \text{ for all } x \in X, \text{ and } f(x) = 0 \text{ if } x \in X \setminus O\}.$$

Then, for any  $A \subset X$ , let

$$\mu^*(A) = \inf\{\mu^*(O) : O \text{ is open, } O \subset X, \text{ and } A \subset O\}$$

Then, we claim the following 5 properties of  $\mu^*$ .

$$\mu^* \text{ is a outer measure on } X \text{ with } \mu^*(A) \leq l(1) \text{ for all } A \subset X. \quad (4.1)$$

$$\mu^*(A) = \inf\{\mu^*(O) : O \text{ is open, } O \subset X, \text{ and } A \subset O\} \quad (4.2)$$

$$\text{If } O \subset X \text{ is open, then } O \text{ is measurable.} \quad (4.3)$$

$$\text{If } U \subset X \text{ is Borel, then } U \text{ is measurable.} \quad (4.4)$$

$$\text{If } A \subset X \text{ is measurable, then } \mu^*(A) = \sup\{\mu^*(K) : K \subset A, K \text{ is compact}\} \quad (4.5)$$

Claim(1) follows from the definition of  $\mu^*$  because Null Empty Set, Subadditivity, and Monotonicity all follow easily from our description of  $\mu^*$ .

Claim(2) is part of the definition of  $\mu^*$ .

Claim(3): We want to show that for any open  $O \subset X$ ,  $\mu^*(A) = \mu^*(A \cap O) + \mu^*(A \cap (X \setminus O))$  for all  $A \subset X$ . The inequality  $\mu^*(A) \leq \mu^*(A \cap O) + \mu^*(A \cap (X \setminus O))$  is trivial by the Monotonicity of

$\mu^*$ . Thus, what left to be shown is that  $\mu^*(A) \geq \mu^*(A \cap O) + \mu^*(A \cap (X \setminus O))$ , or equivalently,  $\mu^*(A) \geq \mu^*(A \cap O) + \mu^*(A \cap (X \setminus O)) - \varepsilon$  for all  $\varepsilon > 0$ .

Suppose  $O \subset X$  is open.

Let  $\varepsilon_1 > 0$ ,  $A \subset X$  be given, then there exists open  $B \subset X$  such that  $\mu^*(A) \geq \mu^*(B) - \varepsilon_1$  and  $A \subset B$ . Then  $\mu^*(B \cap O) \geq \mu^*(A \cap O)$  and  $\mu^*(B \cap (X \setminus O)) \geq \mu^*(A \cap (X \setminus O))$ . Then if  $\mu^*(B) \geq \mu^*(B \cap O) + \mu^*(B \cap (X \setminus O)) - \varepsilon_1$ , we have that  $\mu^*(A) \geq \mu^*(A \cap O) + \mu^*(A \cap (X \setminus O)) - 2\varepsilon_1$ . By adjusting  $\varepsilon_1$  we will have our desired result.

To prove  $\mu^*(B) \geq \mu^*(B \cap O) + \mu^*(B \cap (X \setminus O)) - \varepsilon_1$ , let  $\varepsilon_2 > 0$  and we will construct continuous functions  $f_1, f_2, f_3 : X \rightarrow [0, 1]$  and a open set  $O' \supset X \setminus O$ , such that:

$$\mu^*(B \cap O) \leq l(f_1) + \varepsilon_2, f_1 \text{ nonzero only on } B \cap O$$

$$\mu^*(B \cap (X \setminus O)) \leq l(f_2) + \varepsilon_2, f_2 \text{ nonzero only on } B \cap O'$$

$$f_3 = f_1 + f_2$$

If we are able to construct such  $f_1, f_2, f_3$  and  $O'$ , we will have that  $f_3 : X \rightarrow [0, 1]$  is a continuous function that is nonzero only on  $B$ . Then

$$\mu^*(B) + 2\varepsilon_2 \geq l(f_3) + 2\varepsilon_2 = l(f_1 + f_2) + 2\varepsilon_2 \geq \mu^*(B \cap O) + \mu^*(B \cap (X \setminus O))$$

By adjusting  $\varepsilon_2$ , we can have that  $\mu^*(B) \geq \mu^*(B \cap O) + \mu^*(B \cap (X \setminus O)) - \varepsilon_1$  and we are done.

Now let us construct  $f_1, f_2, f_3$  and  $O'$ , let  $\varepsilon_3 > 0$ . Since  $B \cap O$  is open, by the definition of  $\mu^*$ , we can find a continuous function  $f'_1 : X \rightarrow [0, 1]$  such that  $0 \leq f'_1(x) \leq 1$  for all  $x \in X$ , and  $f'_1(x) \neq 0$  only if  $x \in B \cap O$ . Also,  $\mu^*(B \cap O) \leq l(f'_1) + \varepsilon_3$ . Since  $l$  is a positive linear functional, then  $l(f'_1) \leq l(1) < \infty$ . Now let  $\gamma > 0$  be given such that  $\frac{\gamma}{1+\gamma}l(1) < \varepsilon_3$ . Let  $f_1 = \frac{f'_1}{1+\gamma}$ . Then we have the following inequality:

$$\mu^*(B \cap O) \leq l(f'_1) + \varepsilon_3 = \frac{1}{1+\gamma}l(f'_1) + \frac{\gamma}{1+\gamma}l(f'_1) + \varepsilon_3 \leq \frac{1}{1+\gamma}l(f'_1) + 2\varepsilon_3 = l(f_1) + 2\varepsilon_3$$

Since  $f'_1$  is continuous, then  $O' = f'^{-1}_1((-\infty, \gamma])$  is an open set and  $O' \supset X \setminus (B \cap O) \supset X \setminus O$ . Now,  $O' \cap B$  is a open set and using the definition of  $\mu^*$  again, we can find a continuous function

$f'_2 : X \rightarrow [0, 1]$  such that  $0 \leq f'_2(x) \leq 1$  for all  $x \in X$ , and  $f'_2(x) \neq 0$  only if  $x \in B \cap O'$ . Also,  $\mu^*(B \cap O') \leq l(f'_2) + \varepsilon_3$ . Let  $f_2 = \frac{f'_2}{1+\gamma}$ , and  $f_3 = f_1 + f_2$ . Since  $l$  is a positive linear functional, then  $l(f'_2) \leq l(1) < \infty$ .

Observe that we have the following inequality:

$$\begin{aligned} \mu^*(B \cap (X \setminus O)) &\leq \mu^*(B \cap O') \leq l(f'_2) + \varepsilon_3 = \frac{1}{1+\gamma}l(f'_2) + \frac{\gamma}{1+\gamma}l(f'_2) + \varepsilon_3 \\ &\leq \frac{1}{1+\gamma}l(f'_2) + 2\varepsilon_3 \\ &= l(f_2) + 2\varepsilon_3 \end{aligned}$$

Again, by adjusting  $\varepsilon_3$  we can make  $f_1, f_2$  have the desired properties.

Now, we need to show that  $0 \leq f_3(x) \leq 1$  for all  $x \in X$ . Since  $f_1 \neq 0$  only on  $B \cap O$  and  $f_2 \neq 0$  only on  $B \cap O'$ . Then on  $B \cap O \cap O' \subset O'$ , we have  $f_3 = f_1 + f_2 = \frac{f'_1}{1+\gamma} + \frac{f'_2}{1+\gamma} \leq \frac{\gamma}{1+\gamma} + \frac{1}{1+\gamma} = 1$ ; on  $(B \cap O) \setminus O'$ , we have  $f_1 \leq 1, f_2 = 0$ ; on  $O' \setminus (B \cap O)$ , we have  $f_1 = 0$  and  $f_2 \leq 1$ . Thus,  $0 \leq f_3(x) \leq 1$  for all  $x \in X$ .

Claim(4) is true because the set of all measurable sets is a  $\sigma$ -algebra, by Claim(3), we know that the set of all measurable sets contains the set of all open sets, since the Borel algebra is the smallest  $\sigma$ -algebra that contains all the open sets in  $X$  then every Borel set is measurable.

Claim(5): Suppose  $A \subset X$  is measurable, then:

$$\begin{aligned} \mu^*(A) &= \mu^*(X) - \mu^*(X \setminus A) = \mu^*(X) - \inf\{ \mu^*(O) : O \text{ is open, } O \subset X, \text{ and } X \setminus A \subset O \} \\ &= \mu^*(X) - \inf\{ \mu^*(X) - \mu^*(X \setminus O) : X \setminus O \text{ is compact, } X \setminus O \subset A \} \\ &= \sup\{ \mu^*(X \setminus O) : X \setminus O \text{ is compact, } X \setminus O \subset A \} \\ &= \sup\{ \mu^*(K) : K \subset A, K \text{ is compact} \} \end{aligned}$$

Now let  $\mu$  be the restriction of  $\mu^*$  to the Borel sets, then  $\mu$  is a measure. Using claim(2),(5), we know that  $\mu$  is regular. Since  $\mu(X) = \mu^*(X) = l(1) < \infty$ , we have that  $\mu$  is finite.  $\mu$  is unique because suppose  $\mu', \mu''$  are regular measures that has the desired property, then since they are regular, they are completely determined by their values on compact sets. Suppose  $K \subset X$

is compact hence Borel and measurable; let  $\varepsilon > 0$ , we have that there exists  $K' \supset K$  such that  $\mu''(K') \leq \mu''(K) + \varepsilon$ . By Urysohn's Lemma, there exists continuous function  $f \in C(X, [0, 1])$  such that  $f(x) = 1$  on  $K$  and  $f(x) = 0$  on  $X \setminus O$  where  $O \supset K'$  is open. Then:

$$\mu'(K) = \int \chi_K(x) d\mu'(x) \leq \int f(x) d\mu'(x) = \int f(x) d\mu''(x) \leq \int \chi_{K'}(x) d\mu''(x) = \mu''(K') \leq \mu''(K) + \varepsilon$$

Then we can conclude that  $\mu'(K) \leq \mu''(K)$ . Now if we apply the same argument with  $\mu', \mu''$  interchanged, we have that  $\mu''(K) \leq \mu'(K)$ . Thus,  $\mu''$  agrees with  $\mu'$  on every compact sets, and hence agree on all of the Borel sets. Therefore,  $\mu$  is unique.

The only thing left to be proved is that  $l(f) = \int f(x) d\mu(x)$  for all  $f \in C(X, \mathbb{R})$ . Notice that it is enough to prove that  $l(f) \leq \int f(x) d\mu(x)$  for all  $f \in C(X, \mathbb{R})$  because the linearity of  $l$  implies that

$$-l(f) = l(-f) \leq \int -f(x) d\mu(x) = - \int f(x) d\mu(x)$$

Thus  $l(f) \leq \int f(x) d\mu(x)$  and  $-l(f) \leq - \int f(x) d\mu(x)$  for all  $f$  will give us the desired result.

Now, Let  $K$  be the support of an arbitrary  $f \in C(X, \mathbb{R})$ , let  $[a, b]$  be the interval that contains the range of  $f$ . Suppose  $\varepsilon_4 > 0$  and we pick  $y_i$  for  $i = 0, 1, \dots, n$  such that  $y_i - y_{i-1} < \varepsilon_4$  and

$$y_0 < a < y_1 < \dots < y_n = b$$

We define  $E_i = f^{-1}((y_{i-1}, y_i]) \cap K$  for  $i = 0, 1, \dots, n$ . Since  $f$  is continuous, then  $f$  is also Borel measurable. Therefore,  $E_i$ 's are disjoint Borel sets and  $\bigcup_{i=1}^n E_i = K$ .

Then, for each  $E_i$ , there exists open  $O_i \supset E_i$  where  $\mu(O_i) \leq \mu(E_i) + \frac{\varepsilon_4}{n}$  and  $f(x) < y_i + \varepsilon_4$  on  $O_i$ .

Define  $h_i(y) = \frac{d(y, X \setminus O_i)}{\sum_{j=0}^n d(y, X \setminus O_j)}$  for all  $i = 0, 1, \dots, n$ . Notice that  $h_i \in C(X, \mathbb{R})$  for all  $i$ ,  $0 \leq h_i \leq 1$ ,  $h_i(x) \neq 0$  only if  $x \in O_i$  and  $\sum_{i=0}^n h_i = 1$ . Then:

$$\begin{aligned} l(f) &= \sum_{i=0}^n l(h_i f) \leq \sum_{i=0}^n (y_i + \varepsilon_4) l(h_i) \leq \sum_{i=0}^n (y_i + \varepsilon_4) \mu(O_i) \leq \sum_{i=0}^n (y_i + \varepsilon_4) \left[ \mu(E_i) + \frac{\varepsilon_4}{n} \right] \\ &\leq \sum_{i=0}^n (y_{i-1}) \mu(E_i) + 2 \sum_{i=0}^n \varepsilon_4 \mu(E_i) + \sum_{i=0}^n (y_i + \varepsilon_4) \frac{\varepsilon_4}{n} \\ &\leq \int f(x) d\mu(x) + 2\varepsilon_4 \mu(K) + \varepsilon_4 b \end{aligned}$$

Adjusting  $\varepsilon_4$  we may conclude that  $l(f) \leq \int f(x) d\mu(x)$  and we are done.  $\square$

# Chapter 5

## Properties of Spectral Radius

### 5.1 Introduction

In this section, we introduce the definitions of spectrum, spectral radius and we examine some interesting properties of spectral radius. We assume basic functional analysis knowledge in this section. Proofs for those assumed knowledge can be found in the book "Introductory Functional Analysis and Applications" by Kreyszig.

### 5.2 Preliminary Definitions

**Definition 5.2.1.**  $B(X, X)$  denotes the space of bounded operators from  $X$  to  $X$

An important fact to remember is that  $B(X, X)$  is complete.

**Definition 5.2.2.** The norm of an operator  $T \in B(X, X)$  is defined as  $\|T\| = \sup\{\frac{\|Tx\|}{\|x\|} : x \in X\}$ .

Suppose  $X$  is a complex normed space and  $T : D(T) \subset X \rightarrow X$  is a linear operator.

Let  $T_\lambda = T - \lambda I$  where  $\lambda$  is a complex number and  $I$  is the identity operator.

If  $T_\lambda$  is injective, then  $T_\lambda$  is a bijection from  $X$  to  $T_\lambda(X)$ . In this case, we let  $R_\lambda(T) : T_\lambda(X) \rightarrow X$  be the inverse map of  $T_\lambda$ . The domain of  $R_\lambda(T)$  is therefore  $T_\lambda(X)$ . For different  $\lambda$ , different situations might happen. We can categorize them into following categories.

**Definition 5.2.3.** (Resolvent set, Spectrum) The resolvent set  $\rho(T)$  of  $T$  is the set of all complex numbers  $\lambda$  such that  $R_\lambda(T)$  is injective;  $R_\lambda(T)$  is bounded; and  $R_\lambda(T)$  is defined on a set which is dense in  $X$ . The spectrum of  $T$  is the set  $\sigma(T) = \mathbb{C} \setminus \rho(T)$ .

**Definition 5.2.4.** (Point Spectrum) The point spectrum  $\sigma_p(T)$  is the set of complex numbers  $\lambda$  such that  $R_\lambda(T)$  is not injective.

**Definition 5.2.5.** (Continuous Spectrum) The point spectrum  $\sigma_c(T)$  is the set of complex numbers  $\lambda$  such that  $R_\lambda(T)$  exists and  $R_\lambda(T)$  is defined on a dense subset of  $X$  but  $R_\lambda(T)$  is unbounded.

**Definition 5.2.6.** (Residual Spectrum) The point spectrum  $\sigma_r(T)$  is the set of complex numbers  $\lambda$  such that  $R_\lambda(T)$  exists but  $R_\lambda(T)$  is defined on a subset of  $X$  that is not dense in  $X$ .

Notice that  $\mathbb{C} = \rho(T) \cup \sigma(T) = \rho(T) \cup \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$ .

**Definition 5.2.7.** (Spectral Radius) The spectral radius  $r(T)$  of an operator  $T \in B(X, X)$  where  $X$  is a complex Banach space is defined as  $r(T) = \sup_{\lambda \in \sigma(T)} |\lambda|$ .

## 5.3 Some Propositions

**Theorem 5.3.1.** Let  $T \in B(X, X)$  where  $X$  is a complex Banach space. if  $\|T\| < 1$ , then  $(I - T)^{-1}$  exists as a bounded linear operator on the whole space  $X$  and  $(I - T)^{-1} = \sum_{i=0}^{\infty} T^i$ .

*Proof.* From basic functional analysis knowledge we know that  $\|T^i\| \leq \|T\|^i$ . Also, since the series  $\sum \|T\|^i$  converges if  $\|T\| < 1$ . Because  $B(X, X)$  is complete, we have that absolute convergence implies convergence. Thus  $\sum_{i=0}^{\infty} T^i$  is a bounded linear operator.

Now observe that  $(I - T)(I + T + \dots + T^n) = I - T^{n+1} = (I + T + \dots + T^n)(I - T)$ . Since  $\sum_{i=0}^{\infty} T^i$  converges, then  $\|T^n\| \leq \|T\|^n \rightarrow 0$  as  $n \rightarrow \infty$ .

Then we have:

$$\lim_{n \rightarrow \infty} (I - T)(I + T + \dots + T^n) = \lim_{n \rightarrow \infty} (I + T + \dots + T^n)(I - T) = \lim_{n \rightarrow \infty} I - T^{n+1} = I$$

Since  $\sum_{i=0}^{\infty} T^i = \lim_{n \rightarrow \infty} (I + T + \dots + T^n)$ . Then  $\sum_{i=0}^{\infty} T^i (I - T) = I = (I - T) \sum_{i=0}^{\infty} T^i$ . Therefore,  $(I - T)^{-1} = \sum_{i=0}^{\infty} T^i$ .  $\square$

**Theorem 5.3.2.** *The set of all invertible operators in  $B(X, X)$  is open.*

*Proof.* Suppose  $T$  is invertible. Consider  $U \in B(X, X)$  such that  $\|U - T\| < \frac{1}{\|T^{-1}\|}$ . Since  $\|I - T^{-1}U\| = \|T^{-1}(T - U)\| \leq \|T^{-1}\| \|U - T\| < 1$  Then we have that  $T^{-1}U$  is invertible by Theorem 3.1. Then, we have that  $U$  is invertible. Since  $T$  is not specified, we have that the set of all invertible operators in  $B(X, X)$  is open.  $\square$

**Theorem 5.3.3.** *The spectrum  $\sigma(T)$  of a operator  $T \in B(X, X)$  where  $X$  is a complex Banach space is a closed set.*

*Proof.* Consider the function  $\phi : \mathbb{C} \rightarrow B(X)$  where  $\phi(\lambda) = T - \lambda I$ . Notice that  $\phi$  is a continuous function. Since the set  $S$  of invertible operators in  $B(X, X)$  is open, then  $\phi^{-1}(S)$  is open. Since  $\phi^{-1}(S)$  is the resolvent of  $T$ , we have that  $\sigma(T)$  is closed.  $\square$

**Theorem 5.3.4.** *The spectrum  $\sigma(T)$  of a operator  $T \in B(X, X)$  where  $X$  is a complex Banach space lies in the disk  $\bar{B}_{\|T\|}(O)$ .*

*Proof.* Let  $\lambda \neq 0$  and  $|\lambda| > \|T\|$ . Then We have that  $\|\frac{1}{\lambda}T\| < 1$ . Notice  $R_{\lambda}(T)$  exists because  $R_{\lambda}(T) = (T - \lambda I)^{-1} = -\frac{1}{\lambda}(I - \frac{1}{\lambda}T)^{-1} = -\frac{1}{\lambda} \sum_{i=0}^{\infty} (\frac{1}{\lambda}T)^i$ . Since the series converges when  $\|\frac{1}{\lambda}T\| < 1$ . Thus, we have that for all  $\lambda$  such that  $|\lambda| > \|T\|$ ,  $\lambda \in \rho(T)$ . Thus,  $\sigma(T) \subset \bar{B}_{\|T\|}(O)$ .  $\square$

Then it is clear that  $r(T) \leq \|T\|$ . As a matter of fact, we can further prove that  $r(T) = \lim_{n \rightarrow \infty} (\|T^n\|)^{\frac{1}{n}}$ . But first, we need more backgrounds.

**Theorem 5.3.5.** *The spectrum  $\sigma(T)$  of an operator  $T \in B(X, X)$  is non-empty*

*Proof.* Firstly, we shall prove that  $R_T : \mathbb{C} \setminus \sigma(T) \rightarrow B(X, X)$  given by  $R_T(\lambda) = (T - \lambda I)^{-1}$  is analytic (its Taylor series about  $\lambda$  converges to the function in some neighborhood for every  $\lambda \in \mathbb{C}$ ).

Suppose  $\lambda_0 \in \mathbb{C} \setminus \sigma(T)$ , then for all  $\lambda$  such that  $|\lambda - \lambda_0| < \frac{1}{\|R_T(\lambda_0)\|}$ , we have that  $\|(\lambda - \lambda_0)R_T(\lambda_0)\| < 1$ . Then use previous theorem,  $I - (\lambda - \lambda_0)R_T(\lambda_0)$  is invertible.

Since  $\lambda_0 \notin \sigma(T)$ ,  $(T - \lambda_0 I)$  is invertible. Notice

$$T - \lambda I = T - \lambda_0 I - (\lambda - \lambda_0)I = (T - \lambda_0 I) \cdot (I - (\lambda - \lambda_0)R_T(\lambda_0))$$

We may see that for all  $\lambda_0 \in \mathbb{C}$   $R_T$  is defined in the open disk of radius  $\frac{1}{\|R_T(\lambda_0)\|}$  centered at  $\lambda_0$

Then  $T - \lambda$  is also invertible and  $(T - \lambda I)^{-1} = R_T(\lambda) = (I - (\lambda - \lambda_0)R_T(\lambda_0))^{-1} \cdot R_T(\lambda_0)$ . Using previous theorem, we have

$$R_T(\lambda) = \left( \sum_{n=0}^{\infty} R_T(\lambda_0)^n (\lambda - \lambda_0)^n \right) \cdot R_T(\lambda_0)$$

Thus  $R_T$  is analytic.

Suppose  $\sigma(T)$  is empty, then  $R_T$  is defined on  $\mathbb{C}$ .

Since  $R_T$  is continuous on the disk  $|\lambda| \leq \|T\|$ , and for  $|\lambda| > \|T\|$ , we have

$$\|R_T(\lambda)\| = \|\lambda^{-1} (I - \frac{T}{\lambda})^{-1}\| \leq \frac{\frac{1}{|\lambda|}}{1 - \frac{1}{|\lambda|} \|T\|}.$$

Thus, as  $|\lambda| \rightarrow \infty$ ,  $\|R_T(\lambda)\| \rightarrow 0$ .

This shows that  $R_T$  is bounded. By Liouville's theorem,  $R_T$  is constant 0. However, this means that  $R_T$  is invertible which is a contradiction.  $\square$

**Theorem 5.3.6.** (*Spectral Mapping Theorem*) Suppose  $p(x) = a_n x^n + \dots + a_0$  is a polynomial, then  $\sigma(p(T)) = p(\sigma(T))$ .

*Proof.* Firstly, we prove that  $\sigma(p(T)) \subset p(\sigma(T))$ .



Suppose  $\lambda \in \sigma(p(T))$ , then  $p(T) - \lambda I$  is not bijective. Since  $p$  is a polynomial, then  $p(x) - \lambda$  is also a polynomial and hence we can factorize the polynomial so that  $p(x) - \lambda = a_n(x - \alpha_1)\dots(x - \alpha_n)$ . Naturally,  $p(T) - \lambda I = a_n(T - \alpha_1)\dots(T - \alpha_n)$ . If  $\alpha_i \in \rho(T)$  for all  $i$ , then  $p(T) - \lambda I$  is a product of invertible operators and hence invertible, which contradict the fact that  $p(T) - \lambda I$  is not bijective. Thus, some  $\alpha_j$  is in  $\sigma(T)$ . Then we have that  $p(\alpha_j) - \lambda = 0$  thus  $\lambda \in p(\sigma(T))$ .

Now we prove that  $\sigma(p(T)) \supset p(\sigma(T))$ .

Suppose  $\lambda \in p(\sigma(T))$ . Then  $\lambda = p(\beta)$  for some  $\beta \in \sigma(T)$ .

Since  $\lambda = p(\beta)$ , we have that  $p(x) - \lambda = (x - \beta)q(x)$  where  $q$  is another polynomial. Then consider  $p(T) - \lambda I = (T - \beta I)q(T)$ .

Since  $\beta \in \sigma(T)$ , we have the following two situations:

If  $T - \beta I$  is not injective; Suppose  $p(T) - \lambda I$  is invertible, then  $I = (p(T) - \lambda I)^{-1}(p(T) - \lambda I) = (p(T) - \lambda I)^{-1}(T - \beta I)q(T) = (T - \beta I)(p(T) - \lambda I)^{-1}q(T) = (p(T) - \lambda I)^{-1}q(T)(T - \beta I)$ . Then  $T - \beta I$  is invertible which is a contradiction. Thus  $p(T) - \lambda I$  is not invertible and so  $\lambda \in \sigma(p(T))$ .

If  $T - \beta I$  is injective, then  $(T - \beta I)^{-1}$  has a domain that is not dense in  $X$ . Therefore, we have that  $(T - \beta I)q(T) = p(T) - \lambda I$  has a range that is not dense in  $X$ . Thus  $\lambda \in \sigma(p(T))$ .

In either cases,  $\sigma(p(T)) \supset p(\sigma(T))$ . Then we have that  $\sigma(p(T)) = p(\sigma(T))$ . □

We are now ready to prove the main theorem of this chapter.

**Theorem 5.3.7.** (Gelfand's Formula) *The spectral radius  $r(T)$  of  $T$  is equal to  $\lim_{n \rightarrow \infty} (\|T^n\|)^{\frac{1}{n}}$ .*

*Proof.* Using the previous theorem, it is clearly true that  $[r(T)]^n \leq r(T^n) \leq \|T^n\|$  for all  $n \in \mathbb{N}$ . Thus,  $r(T) \leq (\|T^n\|)^{\frac{1}{n}}$  for all  $n \in \mathbb{N}$ . Then  $r(T) \leq \liminf_{n \rightarrow \infty} (\|T^n\|)^{\frac{1}{n}} \leq \limsup_{n \rightarrow \infty} (\|T^n\|)^{\frac{1}{n}}$ .

Now, we are going to show that  $\limsup_{n \rightarrow \infty} (\|T^n\|)^{\frac{1}{n}} = r(T)$ . Once we establish the equality, we are done.

From complex analysis, we know that  $\sum c_n z^n$  converges absolutely when  $|z| < r$  where  $r$  is the convergence radius and  $\frac{1}{r} = \limsup (|c_n|)^{\frac{1}{n}}$ .

Consider  $-\frac{1}{z} \sum_{i=0}^{\infty} (\frac{1}{z}T)^i$  where  $\frac{1}{z} \in \mathbb{C}$ . Notice that  $\|\sum_{i=0}^{\infty} (\frac{1}{z}T)^i\| \leq \sum_{i=0}^{\infty} \|T\|^i (\frac{1}{z})^i$ . Then since  $\sum_{i=0}^{\infty} \|T\|^i (\frac{1}{z})^i$  converges absolutely with an convergence radius of  $\frac{1}{\limsup(\|T^n\|)^{\frac{1}{n}}}$ . Therefore, if  $|z| > \limsup(\|T^n\|)^{\frac{1}{n}}$ ,  $\sum_{i=0}^{\infty} \|T\|^i (\frac{1}{z})^i$  converges absolutely and hence  $-\frac{1}{z} \sum_{i=0}^{\infty} (\frac{1}{z}T)^i = (T - zI)^{-1} = R_z(T)$  exists. Then we know that for all  $z$  with  $|z| > \limsup(\|T^n\|)^{\frac{1}{n}}$ ,  $z$  is in the resolvent set of  $T$ . Since  $\limsup(\|T^n\|)^{\frac{1}{n}}$  is the smallest number that makes the above condition happen. Then, in another word, we have that  $\limsup(\|T^n\|)^{\frac{1}{n}} = r(T)$  and we are done. □

For the next result, we restrict our attention to the case where  $X$  is equipped with an inner product  $\langle \cdot, \cdot \rangle$  such that  $\|x\| = \sqrt{\langle x, x \rangle}$  for all  $x \in X$ . That is,  $X$  is assumed to be a Hilbert space.

**Definition 5.3.8.** The adjoint  $T^*$  of an operator  $T$  is a bounded operator such that  $\langle Tx, y \rangle = \langle x, T^*y \rangle$  for all  $x, y \in X$ .

**Definition 5.3.9.** An operator  $T$  is normal if it commutes with its adjoint. ie:  $TT^* = T^*T$ .

**Definition 5.3.10.** An operator  $T$  is Hermitian (self-adjoint) if it is equal to its adjoint. ie:  $T = T^*$ .

**Corollary 5.3.11.** If  $T$  is self-adjoint or normal, then  $r(T) = \|T\|$ .

*Proof.*

$T$  is self-adjoint:

Suppose  $x \in X$  and  $\|x\| = 1$ . Then  $\|Tx\|^2 = \langle Tx, Tx \rangle = \langle T^2x, x \rangle \leq \|T^2x\| \|x\| = \|T^2x\|$ . Since  $\|T\|^2 \geq \|T^2\|$ . Then  $\|T\|^2 = \|T^2\|$ . Then, by induction, it is easy to prove that  $\|T\|^{2^n} = \|T^{2^n}\|$  for all  $n \in \mathbb{N}$ . Since  $r(T) = \lim_{n \rightarrow \infty} (\|T^{2^n}\|)^{\frac{1}{2^n}}$ , then  $r(T) = \lim_{n \rightarrow \infty} (\|T\|^{2^n})^{\frac{1}{2^n}} = \|T\|$ .

$T$  is normal:

Firstly, Since  $\|T^*\| \|T\| \geq \|T^*T\| = \|T\|^2$ , then  $\|T^*\| \geq \|T\|$ . Since  $T = T^{**}$ , then  $\|T\| = \|T^{**}\| \geq \|T^*\|$ . So,  $\|T^*\| = \|T\|$ . Suppose  $x \in X$  and  $\|x\| = 1$ . Notice  $\|T^*Tx\| \|x\| \geq \langle T^*Tx, x \rangle = \langle Tx, Tx \rangle = \|Tx\|^2$ , then  $\|T^*T\| \geq \|T\|^2$ . But  $\|T^*T\| \leq \|T\| \|T^*\| = \|T\|^2$ . Then  $\|T^*T\| = \|T\|^2$ . Since  $T$  is normal,  $T^*T$  is self-adjoint, so  $r(T^*T) = \|T^*T\| = \|T\|^2 = r(T)^2$ . □

# Chapter 6

## One Good Way to Interpret Spectral Theorem

### 6.1 Introduction

In linear algebra, there is a very important theorem which says that every Hermitian matrix is unitarily diagonalizable. The spectral theorem we will be discussing is a generalized version of that theorem. Despite the fact that some of the concepts need new definitions in infinite dimensional(Hilbert Space) scenario to match their finite dimensional counterparts, the spectral theorem basically says that every Hermitian operator on a Hilbert space is unitarily "diagonalizable". However, since the spectral theorem deals with linear operators on Hilbert spaces, the formulation of the theorem becomes somewhat elusive and difficult to understand. In this section, we will talk about an approach, suggested by Paul Halmos, to understand what does the spectral theorem says using concepts that are relatively elementary compared to other canonical approaches. Halmos believes that the spectral theorem is feared by most people because its statement does not easily give you the finite dimensional cases result. In order to incorporate the general cases, the description of the spectral theorem is hard for us to formulate a intuition or analog to something that we can easily understand. The approach that we are going to introduce comes from "multiplicity theory". Be-

cause this formulation of the theorem only uses elementary measure theory concepts, says Halmos, it makes this subject more accessible to the general audience.

## 6.2 Preliminary Knowledge

In this chapter, the underlying space is always assumed to be a Hilbert space. And the operators are always assumed to be bounded.

**Definition 6.2.1.** An operator  $A$  is normal if it commutes with its adjoint. ie:  $AA^* = A^*A$ .

**Definition 6.2.2.** An operator  $A$  is Hermitian (self-adjoint) if it is equal to its adjoint. ie:  $A = A^*$ .

**Definition 6.2.3.** Suppose  $\phi$  is a complex valued bounded measurable function on a measure space  $X$  with measure  $\mu$ ; suppose an operator  $A$  defined on the Hilbert space  $L^2(\mu)$  by  $(Af)(x) = \phi(x)f(x)$  where  $x \in X$ , then the operator  $A$  is called the multiplication induced by  $\phi$ .

Using basic properties of Hilbert space, it can be easily shown that the adjoint  $A^*$  of  $A$  is induced by  $\bar{\phi}$  which is the complex conjugate of  $\phi$ ; and if  $\psi$  is also a complex valued bounded measurable function on the measure space  $X$ , and  $B$  is induced by  $\psi$ , then the multiplication induced by  $\phi\psi$  is  $AB$ . Then we can see that a multiplication is always normal.

**Definition 6.2.4.** A cyclic vector  $v$  for an Hermitian operator  $A$  on a Hilbert space  $H$  is a vector such that  $S = \{[p(A)]v : p \text{ is a polynomial with complex coefficient} \}$  is dense in  $H$ .

**Theorem 6.2.5.** Let  $T$  be a Hermitian operator on a Hilbert space  $H$ .  $H$  is always the direct sum of a family of subspaces such that the restriction of  $T$  to each of them has a cyclic vector.

*Proof.* Firstly notice that for all  $v \in H$ ,  $S_v = \{[p(T)]v : p \text{ is a polynomial with complex coefficient} \}$  is always a vector space. Now we consider the closed subspace  $\overline{S_v}$ . We can decompose  $H$  as  $H = \overline{S_v} \oplus \overline{S_v}^\perp$ . We define a partial order on subsets of  $H$ . We say  $A \leq B \leq H$  when the following conditions happen:  $A \subset B$  and if  $v \in B, v \notin A$  then  $\overline{S_v} \perp (\bigoplus_{w \in A} \overline{S_w})$  (Notice  $\bigoplus_{w \in A} \overline{S_w} = \overline{\bigoplus_{w \in A} S_w}$  because of the orthogonality). Suppose now we have a chain of subsets  $C = \{A_i : i \in \Lambda\}$  with

respect to the partial order. We will show that  $C$  has an upper bound. Consider  $A = \bigcup_{i \in \Lambda} A_i$ . Suppose  $A_j \in C$ , and  $v \in A$ . Then  $v \in A_l$  for some  $l \in \Lambda$  and  $A_j \leq A_l$ . Then  $\overline{S_v} \perp (\bigoplus_{w \in A_j} \overline{S_w})$  by the way the partial order is defined. Then  $A_j \leq A$  is also true by the definition of the partial order. Thus we have that  $A$  is an upper bound for  $C$ . Then use Zorn's lemma, we have that there is a maximum subset  $M$  of  $H$  with respect to the partial order. We then want to show that  $H = \bigoplus_{v \in W} \overline{S_v}$ . Suppose on the contrary, then let  $v' \in H$  but  $v' \in (\bigoplus_{v \in W} \overline{S_v})^\perp$ . Then observe that  $\bigoplus_{v \in M} \overline{S_v}$  is invariant under  $T$ , and  $T$  is Hermitian, we have that  $(\bigoplus_{v \in W} \overline{S_v})^\perp$  is also invariant under  $T$ . Then  $\overline{S_{v'}} \perp \bigoplus_{v \in M} \overline{S_v}$  as a result. Then  $M \cup \{v'\} \geq M$  with respect to the partial order. This contradicts the maximality of  $M$ . Thus  $H = \bigoplus_{v \in W} \overline{S_v}$  and we are done.  $\square$

### 6.3 Spectral Theorem

**Theorem 6.3.1.** *If  $A$  is a Hermitian operator on a Hilbert space  $H$ , then there exists a bounded measurable function  $\phi$  on some measure space  $X$  with measure  $\mu$ , and there exists an isometry  $U : L^2(\mu) \rightarrow H$  such that  $(U^{-1}AUf)(x) = \phi(x)f(x)$  where  $x \in X$ .*

Note: For example, if  $\mu$  is the discrete measure on a finite set. Then  $L^2(\mu) = \mathbb{R}^n$ , hence  $f : [n] \rightarrow \mathbb{R}$  is a vector  $v$  such that  $v_i = f(i)$ . In that case, it is just the linear algebra version of the spectral theorem (i.e. Every Hermitian matrix is unitarily diagonalizable)

In general, notice that the theorem conveys the idea that we can modify  $A$  unitarily so that  $A$  can be seen as a multiplication induced by a measurable function  $\phi$  on a measure space  $X$ . This is a generalization of the finite dimensional case. If  $A$  is a  $n \times n$  Hermitian matrix, then the measure space  $X$  is going to be the finite set with  $n$  element with the counting measure.  $L^2(\mu)$  would therefore be the  $n$ -dimensional Euclidean space.  $\phi(x)$  will equal to one of the eigenvalues of  $A$  for each  $x \in X$ . We are all familiar with another way of putting this result - "A is unitarily equivalent to some diagonal matrix  $B$  and the eigenvalues of  $A$  appear in the diagonal of  $B$ ." Although both cases are illustrating the same idea, the generalized case is much harder to prove. We will make use all of the results from previous sections in the following proof.

*Proof.* We will be using three major results from previous sections in the proof.

- (Spectral Radius) If  $A$  is Hermitian, then the spectral radius  $r(A) = \|A\|$ .
- (The Riesz representation theorem) If  $L : C(K, \mathbb{R}) \rightarrow \mathbb{R}$  is a positive linear functional where  $K$  is compact, then there is a unique finite Borel measure  $\mu$  on  $K$  such that  $L(f) = \int f d\mu$ .
- (The Stone-Weierstrass theorem) Every continuous function  $f : K \rightarrow \mathbb{R}$  ( $K$  is compact) is the uniform limit of polynomials on  $K$ .

Using previous theorem, we know that  $H$  can be written as a direct sum  $\bigoplus S_v$  of subspaces such that the restriction of  $A$  to any of the subspaces has a cyclic vector. Notice that if we can prove spectral theorem for all of the restrictions, we have that for each subspace, there is a measure space  $X_v$  and a measure  $\mu_v$  that satisfy the result of spectral theorem. Then  $\bigoplus X_v$  and  $\bigoplus \mu_v$  will satisfy the spectral theorem for the entire space  $H$ . Therefore, WLOG, we can assume  $A$  has a cyclic vector  $v$  on  $H$  and continue the proof from here.

Now, for each real polynomials  $p$  consider  $L(p) = \langle p(A)v, v \rangle$ . Notice that  $L$  is a functional on the space of real polynomials of the linear behavior of inner product. Observe that, using properties of spectral radius for a Hermitian operator:

$$\begin{aligned}
 |\langle p(A)v, v \rangle| &\leq \|p(A)\| \|v\|^2 = r(p(A)) \|v\|^2 \\
 &= \sup\{|\lambda| : \lambda \in \sigma(p(A))\} \|v\|^2 \\
 &= \sup\{|p(\lambda)| : \lambda \in \sigma(A)\} \|v\|^2 \\
 &= \|p\|_{\sigma(A)} \|v\|^2
 \end{aligned}$$

Thus  $L$  is bounded. Since  $\sigma(A)$  is compact, then the space of polynomials on  $\sigma(A)$  is dense in the space  $C(\sigma(A), \mathbb{R})$ . We have that  $L$  can be extended to  $C(\sigma(A), \mathbb{R})$ . Let us now prove  $L$  is positive. If  $f$  is a positive real continuous function, then for all  $\varepsilon > 0$ . Let  $p$  be a polynomial such that  $\|p^2 - f\| < \varepsilon$ . Since  $L(p^2) = \|p(A)v\|^2 \geq 0$  and  $\|L(f) - \langle p(A)^2 v, v \rangle\| < c \|f - p^2\|$  for some constant  $c$  as  $L$  is bounded. We have that  $\|L(f) - \langle p(A)^2 v, v \rangle\| < c\varepsilon$  for all  $\varepsilon > 0$ . Thus  $L(f) \geq 0$ .

Thus, we apply Riesz representation theorem and we have that there exists a finite measure  $\mu$  such that  $L(p) = \int p d\mu$  for every real polynomial  $p$ .

Now suppose  $q$  is an arbitrary polynomial. Let  $Uq = q(A)v$ . Since  $A$  is Hermitian, we have that  $(q(A))^* = \bar{q}(A)$  (We can see this by observing  $\langle \alpha Ax, y \rangle = \langle x, \bar{\alpha} A^* y \rangle = \langle x, \bar{\alpha} Ay \rangle$ ). And  $(q(A))^* q(A) = r^2(A) + s^2(A) = |q|^2(A)$  where  $q(x) = r(x) + is(x)$ . Then we have the following:

$$\int |q|^2 d\mu = \langle \bar{q}(A)q(A)v, v \rangle = \langle (q(A))^* q(A)v, v \rangle = \|q(A)v\|^2 = \|Uq\|^2$$

Thus, we have that  $U$  is an isometry from a dense subset of  $L^2(\mu)$  into  $H$ . Then,  $U$  has a unique isometric extension from  $L^2(\mu)$  into  $H$ . Since  $v$  is a cyclic vector,  $Ran(U)$  is dense in  $H$ . Thus, we have that  $Ran(U) = H$ .

Now let us prove the  $U^{-1}AU$  is a multiplication. Consider  $\phi(\lambda) = \lambda$  where  $\lambda \in \sigma(A)$ , We want to show that  $U^{-1}AU$  is just the multiplication induced by  $\phi$ . Given polynomial  $q$ , let  $q'(x) = xq(x)$  ( $= \phi(x)q(x)$  if  $x \in \sigma(A)$ ). Now, we have  $U^{-1}AUq = U^{-1}Aq(A)v = U^{-1}q'(A)v = U^{-1}Uq' = q'$ . Thus,  $U^{-1}AU$  is equal to the multiplication induced by  $\phi$  on polynomials on  $\sigma(A)$ . Then  $U^{-1}AU$  equals the multiplication induced by  $\phi$  on  $L^2(\mu)$ . □

# Chapter 7

## Banach Algebra and Spectral Theory

### 7.1 Introduction

We can generalize spectral theory into a even more abstract level. The following section is trying to extract ideas corresponding to the ideas from previous sections but with purely Banach Algebra techniques. We can define spectrum as well as other terms appeared in the previous sections within only Banach Algebra context and we will also show Banach Algebra version of the spectral theorem. For simplicity, we will only consider commutative algebra over  $\mathbb{C}$  with identity.

### 7.2 Preliminary Knowledges

Note: The definition of algebra has been given in previous sections. We also assume commutativity in this section.

**Definition 7.2.1.** (Banach Algebra) A Banach Algebra  $A$  is an algebra over the field  $\mathbb{C}$  and also a Banach space with a norm  $\|\cdot\|$  such that for all  $a, b \in A$ ,  $\|ab\| \leq \|a\|\|b\|$ .

If  $A$  has an identity  $\mathbb{1}$ , then  $\|\mathbb{1}\| = 1$

If  $A$  has an identity  $\mathbb{1}$ , then the map  $\alpha \rightarrow \alpha\mathbb{1}$  is an isomorphism from  $\mathbb{C}$  into  $A$  and  $\|\alpha\mathbb{1}\| = |\alpha|$ .



**Definition 7.2.2.** If  $A$  is a Banach algebra with identity and  $a \in A$ , the spectrum of  $a$ , denoted by  $\sigma(a)$ , is defined by  $\sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda \mathbb{1} \text{ is not invertible}\}$ . The resolvent set  $\rho(A)$ , therefore, is  $\mathbb{C} \setminus \sigma(a)$ . The spectral radius  $r(a)$  of  $a$  is  $\sup_{\lambda \in \sigma(a)} |\lambda|$ .

At this point, we can reformulate the theorems we have seen in previous sections. The proofs for these theorems are exactly the same as we have seen before except we need to change the notations accordingly.

**Theorem 7.2.3.** *Let  $a \in A$  where  $A$  is a Banach algebra. if  $\|a\| < 1$ , then  $(1 - a)^{-1}$  exists and  $(1 - a)^{-1} = \sum_{i=0}^{\infty} a^i$ .*

**Theorem 7.2.4.** *The set of all invertible elements in  $A$  is open.*

**Theorem 7.2.5.** *The spectrum  $\sigma(a)$  of an element  $a \in A$  where  $A$  is a Banach algebra is a closed set.*

**Theorem 7.2.6.** *The spectrum  $\sigma(a)$  of an element  $a \in A$  where  $A$  is a Banach algebra lies in the disk  $\bar{B}_{\|a\|}(0) \subset \mathbb{C}$ .*

**Theorem 7.2.7.** *The spectrum  $\sigma(a)$  of an element  $a \in A$  is non-empty*

**Theorem 7.2.8.** *(Spectral Mapping Theorem) Suppose  $p(x) = a_n x^n + \dots + a_0$  is a polynomial, then  $\sigma(p(a)) = p(\sigma(a))$ , where  $a \in A$ .*

**Theorem 7.2.9.** *(Gelfand's Formula) The spectral radius  $r(a)$  of  $a$  is equal to  $\lim_{n \rightarrow \infty} (\|a^n\|)^{\frac{1}{n}}$ .*

**Theorem 7.2.10.** *If  $a$  is Hermitian, then  $r(a) = \|a\|$ .*

## 7.3 Basic Properties for Banach Algebra

Apart from the old theorems, there are also new information we can derive under the Banach algebra settings. We will gradually build the Banach algebra approach to spectral theory and try to recover similar results that we saw in the previous section. In particular, the spectral theorem, under Banach Algebra context, will be examined.

**Theorem 7.3.1.** (The Gelfand-Mazur Theorem). *If  $A$  is a Banach algebra that is also a division ring, then  $A = \mathbb{C}$ , or more precisely  $A = \{\alpha \mathbb{1} : \alpha \in \mathbb{C}\}$ .*

A division ring means that every nonzero element has a multiplicative inverse. Also, from now on, we will abbreviate the notation  $\lambda \mathbb{1}$  to be just  $\lambda$ .

*Proof.* If  $a \in A$ , then  $\sigma(a) \neq \emptyset$ . If  $\lambda \in \sigma(a)$ , then  $a - \lambda$  is not invertible. Then  $a - \lambda = 0$  (ie:  $a = \lambda$ ). □

**Theorem 7.3.2.** *If  $A$  is an abelian Banach algebra, and  $M$  is a maximal ideal, then there is a homomorphism  $h : A \rightarrow \mathbb{C}$  such that  $M = \text{Ker}(h)$ . Conversely, if  $h : A \rightarrow \mathbb{C}$  is a nonzero homomorphism, then  $\text{Ker}(h)$  is a maximal ideal. Moreover, the correspondence  $h \rightarrow \text{Ker}(h)$  is one-to-one and onto.*

Before we start the proof, one interesting observation is that the homomorphisms from  $A$  to  $\mathbb{C}$  can be seen as a special type of functionals.

*Proof.* If  $M$  is a maximal ideal, then I claim that  $M$  is closed.  $M$  is a proper ideal means that  $M$  doesn't include any invertible elements. Thus  $M \subset A \setminus G$  where  $G$  is the set of all invertible elements. We have shown that  $G$  is open in previous section. Thus, we have that  $A \setminus G$  is closed. Then the closure  $cl(M) \subset A \setminus G \neq A$ . Now,  $cl(M)$  is also an ideal because suppose  $x \in cl(M)$ , then there exists  $x_n \in M \rightarrow x$ . Then, for all  $y \in A$ ,  $x_n y \rightarrow xy$  and since  $x_n y \in M$ , we have  $xy \in cl(M)$ . Now, since  $cl(M)$  is a proper ideal and  $M \subset cl(M)$  where  $M$  is maximal, we have that  $M = cl(M)$ . Thus  $M$  is closed.

Since  $M$  is a closed subspace of  $A$ , then  $A/M$  is a Banach space where the norm is defined to be  $\|x + M\| = \inf_{s \in M} \|x + s\|$ .  $A/M$  is also an algebra. In particular,  $A/M$  is actually a Banach algebra. The reason is as follow: Suppose  $x, y \in A$  and  $s, t \in M$ , then  $(x + s)(y + t) = xy + (xt + st + sy) \in xy + M$ . Hence,  $\|(x + M)(y + M)\| = \|(xy + M)\| \leq \|(x + s)(y + t)\| \leq \|x + s\| \|y + t\|$ . Now, if we take the infimum of  $s, t$  over all  $M$ , we have that  $\|(x + M)(y + M)\| \leq \|x + M\| \|y + M\|$ .

Now, let  $\pi : A \rightarrow A/M$  be the natural map ( $\pi(x) = x + M$ ). If  $a \in A$  and  $\pi(a)$  is not invertible in  $A/M$ , then  $\pi(aA) = \{\pi(ax) : x \in A\} = \pi(a)(A/M)$  is an ideal in  $A/M$  that is proper. Consider  $I = \{b \in A : \pi(b) \in \pi(aA)\}$  which is the preimage of  $\pi(aA)$ . We can see that  $I$  is a proper ideal in  $A$ . Notice that  $M \subset I$  because if  $x \in M$ , then  $\pi(x) = (0 + M) \in \pi(aA)$ . Since  $M$  is maximal, then  $M = I$ . Then  $\pi(aA) \subset \pi(I) = \pi(M) = 0$ . Thus, we have  $\pi(a) = 0$ , this implies that  $A/M$  doesn't have any non-invertible element (ie: it is a field). Then we have that  $A/M = \mathbb{C}$  by Gelfand-Mazur theorem. Now, let  $g : A/M \rightarrow \mathbb{C}$  given by  $g(\lambda + M) = \lambda$ , let  $h : A \rightarrow \mathbb{C}$  be given by  $h = g \circ \pi$ . Then  $h$  is a homomorphism and  $\text{Ker}(h) = M$ .

Now, suppose  $h : A \rightarrow \mathbb{C}$  is a nonzero homomorphism. Then  $\text{Ker}(h)$  is a nontrivial ideal. Then  $A/\text{Ker}(h)$  is isomorphic to  $\mathbb{C}$  by the first isomorphism theorem. So  $\text{Ker}(h)$  is maximal.

If  $h, g$  are two nonzero homomorphisms and  $\text{Ker}(h) = \text{Ker}(g) = V$ , then  $h = \lambda g$  for some  $\lambda \in \mathbb{C}$ . Since  $h(1) = 1 = \lambda g(1) = \lambda$ , we have  $h = g$ . This is because suppose  $h, g$  are both homomorphisms from  $A$  to  $\mathbb{C}$ , then  $A/V$  is isomorphic to  $\mathbb{C}$ . Now suppose  $h', g' : A/V \rightarrow \mathbb{C}$  where  $h(x) = h'(x+V)$ ,  $g(x) = g'(x+V)$ . Now suppose  $a \in A$  and  $a+V \in A/V$  such that  $h'(a+V) = 1$ , then  $g'(a+V) = k$  for some  $k \in \mathbb{C}$ . Since  $A/V$  is isomorphic to  $\mathbb{C}$  which is one dimensional, then for all  $x \in A$ ,  $x = \alpha a + x'$  for some  $\alpha \in \mathbb{C}$  and  $x' \in V$ . Then

$$h(x) = h(\alpha a) = h'(\alpha a + V) = \alpha h'(a + V) = \alpha \frac{1}{k} g'(a + V) = \frac{1}{k} g'(\alpha a + V) = \frac{1}{k} g(\alpha a) = \frac{1}{k} g(x)$$

□

**Theorem 7.3.3.** *If  $A$  is abelian and  $g : A \rightarrow \mathbb{C}$  is a nonzero homomorphism, then  $\|g\| = 1$ .*

*Proof.* Let  $a \in A$ . Let  $\lambda = h(a)$ . Suppose  $|\lambda| > \|a\|$ . Then, we have that  $1 - \frac{a}{\lambda}$  is invertible. Then:

$$\begin{aligned} 1 = h(1) &= h\left(\left(1 - \frac{a}{\lambda}\right)\left(1 - \frac{a}{\lambda}\right)^{-1}\right) \\ &= h\left(\left(1 - \frac{a}{\lambda}\right)^{-1}\right) - h\left(\left(1 - \frac{a}{\lambda}\right)^{-1}\right) \frac{h(a)}{\lambda} \\ &= h\left(\left(1 - \frac{a}{\lambda}\right)^{-1}\right) - h\left(\left(1 - \frac{a}{\lambda}\right)^{-1}\right) \cdot 1 = 0 \end{aligned}$$

which is a contradiction. Thus,  $|\lambda| \leq \|a\|$ , so  $\|h\| \leq 1$ . Since  $h(1) = 1$ , then  $\|h\| = 1$ . □

**Definition 7.3.4.** If  $A$  is an abelian Banach algebra, let  $\Sigma = \{ \text{all nonzero homomorphisms from } A \text{ to } \mathbb{C} \}$ . Give  $\Sigma$  the relative weak\* topology associated with  $A^*$  (The dual space of  $A$ ).  $\Sigma$  is called the maximal ideal space.

**Theorem 7.3.5.** *If  $A$  is a abelian Banach algebra, then  $\Sigma$  is a compact Hausdorff space. Moreover, if  $a \in A$ , then  $\sigma(a) = \{h(a) : h \in \Sigma\}$ .*

*Proof.* Using previous theorem, we have  $\Sigma$  is in the unit ball of  $A^*$  which is weak\* compact by the Banach–Alaoglu theorem. Therefore, we only need to prove that  $\Sigma$  is weak\* closed. Let  $h_n \in$  the unit ball of  $A^*$  and  $h_n \rightarrow h$  weakly. For  $a, b \in A$ ,  $h(ab) = \lim_{n \rightarrow \infty} h_n(ab) = \lim_{n \rightarrow \infty} h_n(a)h_n(b) = h(a)h(b)$ . Thus  $h$  is a homomorphism. Since  $h(1) = \lim_{n \rightarrow \infty} h_n(1) = 1$ ,  $h \in \Sigma$ . Thus  $\Sigma$  is a closed subset of a compact set, hence  $\Sigma$  is compact.

If  $h \in \Sigma$  and  $\lambda = h(a)$ , then  $h(a - \lambda) = 0$ . This means that  $a - \lambda$  is not invertible, thus,  $\lambda \in \sigma(a)$ . Thus  $\{h(a) : h \in \Sigma\} \subset \sigma(a)$ . Now suppose  $\lambda \in \sigma(a)$ , then  $a - \lambda$  is not invertible. Then  $(a - \lambda)A$  is a proper ideal. Let  $M$  be a maximal ideal that contains  $(a - \lambda)A$ . There is some  $h \in \Sigma$  such that  $\text{Ker}(h) = M$ , then  $h(a - \lambda) = 0 = h(a) - \lambda$ . Thus  $\sigma(a) \in \{h(a) : h \in \Sigma\}$ .  $\square$

**Definition 7.3.6.** Let  $A$  be an abelian Banach algebra with  $\Sigma$  as maximal ideal space. If  $a \in A$ , then the Gelfand transform of  $a$  is given by  $\hat{a} : \Sigma \rightarrow \mathbb{C}$  where  $\hat{a}(h) = h(a)$ .

**Theorem 7.3.7.** *If  $A$  is an abelian Banach algebra with maximal ideal space  $\Sigma$  and  $a \in A$ , then  $\hat{a}$  is continuous on  $\Sigma$ . The map that takes  $a$  to  $\hat{a}$  is a continuous homomorphism of  $A$  into  $C(\Sigma)$  with norm 1 and the kernel of the map is  $\bigcap \{M : M \text{ is a maximal ideal of } A\}$ . Moreover,  $\|\hat{a}\|_\infty = \lim_{n \rightarrow \infty} \|a\|_n^{\frac{1}{n}}$ .*

*Proof.* We first prove that  $\hat{a}$  is continuous. Suppose  $h_n \rightarrow h$  in the maximal ideal space. Then, by definition,  $h_n \rightarrow h$  weakly in the dual space of  $A$ . Then given  $a \in A$ ,  $\hat{a}(h_n) = h_n(a) \rightarrow h(a) = \hat{a}(h)$ . Thus  $\hat{a}$  is continuous.

Now consider the map  $\phi : A \rightarrow C(\Sigma)$  given by  $\phi(a) = \hat{a}$ . We want to show that  $\phi$  is a continuous homomorphism and  $\|\phi\| = 1$ . Suppose  $a, b \in A$ , then for all  $h \in \Sigma$   $\phi(ab)(h) = \hat{a}b(h) = h(ab) = h(a)h(b) = \hat{a}(h)\hat{b}(h) = \phi(a)(h)\phi(b)(h)$ . Thus,  $\phi$  is a homomorphism. Since for all  $h \in \Sigma$ ,  $|h(a)| =$

$|\hat{a}(h)| \leq \|a\|$  by previous theorem. Thus  $\|\hat{a}\|_\infty \leq \|a\|$ . Since  $\phi(a) = \hat{a}$ , we have that  $\|\phi\| \leq 1$ . Since  $\phi(1) = 1$  as  $\phi$  is a homomorphism,  $\|\phi\| = 1$ .

Observe that  $a \in \text{Ker}(\phi)$  means that  $\hat{a} = 0$ . Equivalently,  $h(a) = 0$  for all  $h \in \Sigma$ . Thus, using previous theorem we have  $a \in \bigcap \{M : M \text{ is a maximal ideal of } A\}$ .

At last, since  $\|\hat{a}\|_\infty = \sup\{|h(a)| : h \in \Sigma\}$ . Use the previous theorem we have that  $\|\hat{a}\|_\infty = \sup\{|\lambda| : \lambda \in \sigma(a)\}$ . Since  $\sup\{|\lambda| : \lambda \in \sigma(a)\} = r(a) = \lim_{n \rightarrow \infty} \|a\|^{\frac{1}{n}}$  as we have seen from previous section, we are done.  $\square$

## 7.4 $C^*$ -algebra and the Spectral Theorem

We will now talk about a specific type of Banach Algebra which is called the  $C^*$ -algebra. In addition to the axioms for general Banach Algebra,  $C^*$ -algebra is also trying to generalize the concept of "Adjoint" we see in the previous sections. At the end of this section, we will see another proof of the Spectral Theorem except that it will be presented in the language of  $C^*$ -algebra.

**Definition 7.4.1.** (Involution) If  $A$  is a Banach algebra, an involution is a map  $a \rightarrow a^*$  from  $A$  to  $A$  such that we have the following properties for  $a, b \in A$  and  $\lambda \in \mathbb{C}$ .

1.  $(a^*)^* = a$
2.  $(ab)^* = b^*a^*$
3.  $(\lambda a + b)^* = \bar{\lambda}a^* + b^*$

**Proposition 7.4.2.** From the above definition, we can conclude that  $1^* = 1$ .

*Proof.* For all  $a \in A$ ,  $1^*a = (1^*a)^{**} = (a^*1)^* = (a^*)^* = a$ . Similarly  $a1^* = a$ , thus  $1^* = 1$ .  $\square$

**Definition 7.4.3.** ( $C^*$ -algebra) A  $C^*$ -algebra is a Banach algebra  $A$  with an involution such that  $\|a^*a\| = \|a\|^2$  for all  $a \in A$ .

**Proposition 7.4.4.**  $\|a^*\| = \|a\|$  for all  $a \in A$  where  $A$  is a  $C^*$ -algebra.

*Proof.* Since  $\|a^*\| \|a\| \geq \|a^*a\| = \|a\|^2$ , then  $\|a^*\| \geq \|a\|$ . Since  $a = a^{**}$ , then  $\|a\| = \|a^{**}\| \geq \|a^*\|$ . So,  $\|a^*\| = \|a\|$ .  $\square$

**Proposition 7.4.5.**  $a = x + iy$  where  $x, y \in A$  are Hermitian for all  $a \in A$  where  $A$  is a  $C^*$ -algebra.

*Proof.* Observe  $a = \frac{a+a^*}{2} + i\left(\frac{ia^*-ia}{2}\right)$ . Notice  $\left(\frac{a+a^*}{2}\right)^* = \frac{a^*+a}{2}$  and  $\left(\frac{ia^*-ia}{2}\right)^* = \frac{-ia+ia^*}{2}$ . Thus  $\frac{a+a^*}{2}$  and  $\frac{ia^*-ia}{2}$  are Hermitian.  $\square$

**Proposition 7.4.6.** For a nonzero homomorphism  $h : A \rightarrow \mathbb{C}$ ,  $h(a^*) = \overline{h(a)}$  for all  $a \in A$  where  $A$  is a  $C^*$ -algebra.

*Proof.* Suppose  $b \in A$  is Hermitian and  $t \in \mathbb{R}$ . Since  $h$  is a nonzero homomorphism,  $\|h\| = 1$  as we proved before. Then

$$\begin{aligned} |h(b+it)|^2 &\leq \|b+it\|^2 = \|(b+it)^*(b+it)\| \\ &= \|(b-it)(b+it)\| \\ &= \|b^2+t^2\| \leq \|b^2\| + t^2 \end{aligned}$$

Since  $h$  is a homomorphism, then  $h(1) = 1$ , so  $h(it) = it$ .

Now if  $h(b) = \alpha + i\beta$  where  $\alpha, \beta \in \mathbb{R}$ , then  $\|b^2\| + t^2 \geq |\alpha + i\beta + it|^2 = \alpha^2 + \beta^2 + 2\beta t + t^2$ .

Since the above inequality holds for all  $t \in \mathbb{R}$ . Then we have that  $2\beta t = 0$  must be true, thus  $\beta = 0$ . Hence  $h(b) \in \mathbb{R}$ . Then  $h(b) = h(b^*) = \overline{h(b)}$  holds.

Now suppose  $a \in A$  is an arbitrary element. Then  $a = x + iy$  where  $x, y$  are Hermitian. Since  $a^* = (x + iy)^* = x - iy$ . Then using the result for Hermitian elements, we have  $h(a^*) = \overline{h(a)}$  as desired.  $\square$

**Theorem 7.4.7.** If  $A$  is an abelian  $C^*$ -algebra and  $\Sigma$  is the maximal ideal space, then the Gelfand transform  $\phi : A \rightarrow C(\Sigma)$  is an isometric conjugate-isomorphism of  $A$  onto  $C(\Sigma)$ .

*Proof.* We know from the previous theorem that for all  $a \in A$ ,  $\|\hat{a}\|_\infty \leq \|a\|$ . Since  $\|\hat{a}\|_\infty$  is also the spectral radius of  $a$ , we have that  $\|\hat{a}\|_\infty = \|a\|$  if  $a$  is Hermitian by the last theorem in the spectral radius section. Notice that  $(a^*a)^* = a^*a$  for all  $a \in A$ . Thus  $a^*a$  is Hermitian and therefore,  $\|\widehat{a^*a}\|_\infty = \|a^*a\|$  for all  $a \in A$ .

Now, suppose  $h \in \Sigma$  and  $a \in A$ . Notice  $\phi(a^*)(h) = \hat{a}^*(h) = h(a^*) = \overline{h(a)} = \overline{\hat{a}(h)} = \overline{\phi(a)(h)}$ . So, we have that  $\phi$  is a conjugate-homomorphism. Since  $\|a\|^2 = \|a^*a\| = \|\widehat{a^*a}\|_\infty = \|\hat{a}^2\|_\infty = \|\hat{a}\|_\infty^2$ . Then  $\|a\| = \|\hat{a}\|_\infty = \|\phi(a)\|$ . Thus  $\phi$  is an isometry.

Since  $A$  is a Banach space, then  $\phi(A)$  is closed. We will now show that  $\phi(A)$  is dense and we are done. Recall the Stone-Weierstrass theorem. Since  $\phi(1) = \hat{1} = 1 \in \phi(A)$ , then  $\phi(A)$  is a subalgebra with constants, so  $\phi(A)$  does not vanish at any points. Since  $\overline{\phi(a)} = \phi(a^*)$ , then  $\phi(A)$  is closed under complex conjugation. Suppose  $h_1, h_2 \in \Sigma$  are different homomorphisms, then  $h_1(a) \neq h_2(a)$  for some  $a \in A$ . Then  $\hat{a}(h_1) \neq \hat{a}(h_2)$ , so  $\phi(a) = \hat{a}$  separates  $h_1, h_2$ . Thus  $\phi(A)$  separates points. Apply Stone-Weierstrass theorem we have that  $\phi(A)$  is dense in  $C(\Sigma)$  and we are done.  $\square$

At last, we are going to present one final corollary, this corollary can be viewed as the  $C^*$ -algebra version of the spectral theorem. As we mentioned before, the following corollary is simply a different formulation of the spectral theorem with  $C^*$ -algebra language.

**Corollary 7.4.8.** *If  $A$  is an **abelian**  $C^*$ -algebra singly generated by an element  $a$ , then  $A$  is isometrically isomorphic to  $C(\sigma(a))$ . In particular, the isometrical isomorphism  $\gamma$  maps  $a$  to the identity function on  $C(\sigma(a))$ . ( $\gamma(a) = \mathbf{1}$ .)*

**Definition 7.4.9.**  $A$  is singly generated by  $a$  if  $A = \overline{\{p(a, a^*) : p \text{ is a polynomial}\}}$  (closure with respect to the  $C^*$ -algebra norm).

Note:  $A$  is an **abelian**  $C^*$ -algebra singly generated by an element  $a$  means that  $a^*a = aa^*$ . This is equivalent to say that  $A$  is a  $C^*$ -algebra singly generated by a **normal** element  $a$ . In particular, when  $a$  is Hermitian, the corollary holds.

*Proof.* Suppose  $\Sigma$  is the maximal ideal space and  $\phi : A \rightarrow C(\Sigma)$  is the Gelfand transform. Define  $\tau : \Sigma \rightarrow \sigma(a)$  given by  $\tau(h) = \hat{a}(h) = h(a)$ .  $\tau$  is continuous because if  $h_i \rightarrow h$  in  $\Sigma$ , then  $h_i$  converges

pointwise to  $h$ , so  $\tau(h_i) = h_i(a) \rightarrow h(a) = \tau(h)$ .  $\tau$  is injective because suppose  $h, g \in \Sigma$ , and  $\tau(h) = h(a) = g(a) = \tau(g)$ . Now since  $h, g$  are homomorphisms we can show that  $h(a^n) = h(a)^n = g(a)^n = g(a^n)$ , and similarly  $h((a^*)^n) = g((a^*)^n)$ . Then, we also have that  $h(p(a, a^*)) = g(p(a, a^*))$  for all polynomial  $p$ . Since  $\{p(a, a^*) : p \text{ is a polynomial}\}$  is dense in  $A$ , we have that  $h$  must agree to  $g$  on all of  $A$ . Thus we have that  $\tau$  is injective. We also have that  $\tau$  is surjective and  $\Sigma$  is compact by Theorem 3.5. Thus,  $\tau$  is a continuous bijection from a compact space to another compact space. Hence,  $\tau$  must be a homeomorphism. Now consider  $\gamma : A \rightarrow C(\sigma(a))$  given by  $\gamma(x) = \hat{x} \circ \tau^{-1}$ . Let  $z \in \sigma(a)$ , then  $\gamma(a)(z) = \hat{a} \circ \tau^{-1}(z) = \tau^{-1}(z)(a) = h_z(a)$  where  $h_z(a) = z$  by definition of  $\tau$ . Similarly,  $\gamma(a^*)(z) = \hat{a}^* \circ \tau^{-1}(z) = \tau^{-1}(z)(a^*) = \overline{h_z(a)}$  where  $\overline{h_z(a)} = \bar{z}$ . Then  $\gamma(a) : \sigma(a) \rightarrow \sigma(a)$  is the identity mapping and  $\gamma(a^*) : \sigma(a) \rightarrow \sigma(a)$  is the conjugate mapping. Since  $\tau$  is a homeomorphism, then  $C(\sigma(a)) \cong C(\Sigma)$ . Since  $C(\Sigma) \cong A$  by previous theorem, then  $C(\sigma(a)) \cong A$ . The last step is to prove  $\gamma$  is an isometry. Since the Gelfand transform  $\phi$  is an isometry and  $\tau$  is a homeomorphism, then  $|\gamma(x)(z)| = |(\phi(x)(\tau^{-1}(z)))|$  for all  $z$ , thus  $\|\gamma(x)\|_\infty = \|\phi(x)\|_\infty = \|x\|$ .  $\square$



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